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On the randomization theory of experiments in nested block designs

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Summary

A general randomization model for experiments in nested block designs is presented and conditions are given for obtaining the best linear unbiased estimators under the model. Since the conditions appear to be severely restrictive, a resolution of the overall randomization model into four relevant stratum submodels is considered. Conditions for obtaining the best linear unbiased estimators under these submodels are found. In addition, it is shown under which conditions the best linear unbiased estimator for a treatment parametric function obtainable under a submodel becomes such an estimator under the overall model. Two discussed examples illustrate application of the theory. Finally, some general concluding remarks complete the paper.

1. Introduction and preliminaries

In an earlier paper by Caliński and Kageyama (1991) a randomization model for experiments with one stratum of blocks of experimental units has been presented and discussed from the point of view of the intra-block and inter-block analysis. The aim of the present paper is to extend the model to the experimental situation in which the blocks are further grouped into some superblocks, forming in that way two strata of blocks. Such situations appear quite often in practice, particularly in agricultural experimentation. Common examples are the lattice designs or, more generally, the so-called resolvable incomplete block designs (see, e.g., John, 1987, Section 3.4). While in an ordinary block design (as considered in the original earlier paper) the stratification of the experimental units leads to three strata, of units within blocks, of blocks within the total experimental

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area, and of the total area, in the extended situation to be considered here four strata can be distinguished. These are of units within blocks, of blocks within superblocks, of superblocks within the total experimental area, and of the total area. In the consequence, the extended randomization model has to take into account three instead of two stages of randomization, viz., of units within blocks, of blocks within superblocks and of the latter within the total area.

A similar situation to that just described occurs when in an ordinary block design the plots (units) are divided into two or more sub-plots. Here, again, four strata appear, of sub-plots within the main plots, of the main plots within blocks, of blocks within the total experimental area, and of that total area (see, e.g., Pearce, 1983, Section 6.1). If the randomization is performed independently within each of the first three strata, then virtually the same randomization model as that mentioned above will apply. Thus, although the randomization model to be considered in the present paper is derived in the context of plots grouped into blocks and the latter grouped into superblocks, it can perfectly well be thought as suitable for subplots grouped into main plots and those into blocks.

In both of these situations, the relations between the strata are of nesting type. One can speak in this context of two systems of blocks, one nested in the other, either of blocks nested in the superblocks or main plots nested in the blocks. Following a recent paper by Mejza and Kageyama (1994), who have generalized the concept of nested balanced incomplete block designs originally introduced by Preece (1967), the term "nested block (NB) design" will be adopted for the designs considered in the present paper. For references concerning nested balanced incomplete block designs or certain extensions of them see Mejza and Kageyama (1994). Similarly as in their, also in the present paper nested block designs will be considered in a most general framework.

As in the original paper by Caliński and Kageyama (1991), the discussion of statistical consequences of the derived randomization model will concern mainly the best linear unbiased estimation of treatment parametric functions under the overall model and under the submodels related to the different strata. In the traditional block designs with two strata of blocks, where the superblock stratum is usually composed of single replicates (see, e.g., John, 1987, Table 8.3), only two analyses are of interest, the intra-block and inter-block analysis, the superblock stratum contributing nothing to the estimation of parametric functions. In the present paper, however, a general situation is to be considered, in which the inter-superblock analysis may also be of interest, at least in some cases.

The basic notation and terminology of the present paper follows that of Caliński and Kageyama (1991). As usual, a block design setting out v treatments in b blocks is described by its $v{\times}b$ incidence matrix $\mathbf{N} = \mathbf{\Delta D}'$, where $\mathbf{\Delta}'$ is the $n{\times}v$ design matrix for treatments and $\mathbf{D}'$ is the $n{\times}b$ design matrix for blocks.

Such design will be denoted by D^*. The grouping of blocks of D^* into superblocks is reflected by the partitions $\mathbf{\Delta}' = [\mathbf{\Delta}_1' : \mathbf{\Delta}_2' : \dots : \mathbf{\Delta}_a']$, $\mathbf{D} = \operatorname{diag}[\mathbf{D}_1 : \mathbf{D}_2 : \dots : \mathbf{D}_a]$ and, consequently, $\mathbf{N} = \operatorname{diag}[\mathbf{N}_1 : \mathbf{N}_2 : \dots : \mathbf{N}_a]$, where $\mathbf{\Delta}_h$, $\mathbf{D}_h$ and $\mathbf{N}_h = \mathbf{\Delta}_h \mathbf{D}_h'$ describe a component design confined to superblock h ($h = 1,2,\dots,a$). The resulting design setting out the v treatments in a superblocks is then denoted by D and described by its $v\times a$ incidence matrix $\mathbf{R} = \mathbf{\Delta}\mathbf{G}'$, where $\mathbf{G}'$ is the $n\times a$ design matrix for superblocks of the form

$$\mathbf{G}' = \mathbf{D}' \operatorname{diag}[\mathbf{1}_{b_1} : \mathbf{1}_{b_2} : \dots : \mathbf{1}_{b_a}] = \operatorname{diag}[\mathbf{1}_{n_1} : \mathbf{1}_{n_2} : \dots : \mathbf{1}_{n_a}] ,$$

with b_h denoting the number of blocks in superblock h, and n_h the number of units (plots) in that superblock, i.e. its size;

$$n_h = \mathbf{1}_{b_h}' \mathbf{k}_h, \quad \text{if} \quad \mathbf{k}_h = [k_{1(h)}, k_{2(h)}, \dots, k_{b_h(h)}]'$$

denotes the vector of block sizes within superblock h. The element r_{ih} of $\mathbf{R}$ denotes the number of replications of treatment i in superblock h. The matrix $\mathbf{R}$ can also be written as

$$\mathbf{R} = [\mathbf{r}_1 : \mathbf{r}_2 : \dots : \mathbf{r}_a], \quad \text{where } \mathbf{r}_h = [r_{1h}, r_{2h}, \dots, r_{vh}]' .$$

Evidently, $\mathbf{N}\mathbf{1}_b = \mathbf{R}\mathbf{1}_a = \mathbf{r} = [r_1, r_2, \dots, r_v]'$ is the vector of treatment replications in the whole NB design, as well as in D^* and D, $\mathbf{N}'\mathbf{1}_v = \mathbf{k} = [\mathbf{k}_1', \mathbf{k}_2', \dots, \mathbf{k}_a']'$ is the vector of block sizes in D^*, $\mathbf{R}'\mathbf{1}_v = \mathbf{n} = [n_1, n_2, \dots, n_a]'$ is the vector of superblock sizes in D. Similarly as $\mathbf{r}^\delta = \mathbf{\Delta}\mathbf{\Delta}'$ and $\mathbf{k}^\delta = \mathbf{D}\mathbf{D}'$ are the diagonal matrices of treatment replications and block sizes, respectively, $\mathbf{n}^\delta = \mathbf{G}\mathbf{G}'$ is the diagonal matrix of superblock sizes, i.e. with numbers n_h on the diagonal. The total number of units, or plots, used in the experiment is $n = \mathbf{1}_v'\mathbf{r} = \mathbf{1}_b'\mathbf{k} = \mathbf{1}_a'\mathbf{n}$. For convenience, a component design described by $\mathbf{N}_h$ will be denoted by D_h ($h = 1,2,\dots,a$).

Since it is intended to consider a general case, disconnected designs are also covered. However, because of the above partitioned structure of $\mathbf{N}$, with its submatrices $\mathbf{N}_h$, $h = 1,2,\dots,a$, corresponding naturally to the superblocks, the assumption made in Caliński and Kageyama (1991, p. 98), that $\mathbf{N}$ has a quasi-diagonal structure reflecting the disconnectedness of the design, cannot be adopted here. Nevertheless, the incidence matrix of a disconnected design D^* or D can always be visualized in that form after proper reordering of its rows and columns.

Similarly as in Caliński and Kageyama (1991), distinction will be made between the potential (or available) number of superblocks, N_A , and the number, a, of those actually chosen from them to the experiment. The usual situation is that $a = N_A$, but in general $a \le N_A$.

Also, it will be convenient to distinct between the potential (available) number of blocks within a superblock, denoting it by B with appropriate subscript, and the number of those of them actually chosen to the experiment, denoting it by b with a relevant subscript. Finally, a distinction will be made between the potential number of units (plots) within a block, denoting it by K with a subscript, and the number of those of them actually used in the experiment, denoting it by k with a subscript.

2. A randomization model

The same approach as that used by Caliński and Kageyama (1991, Section 2) will be adopted here to derive and investigate the model of the variables observed on the n units actually used in the experiment. The extension consists in applying a threefold instead of twofold randomization of superblocks within a total area of them, of blocks within the superblocks, and of units within the blocks.

2.1. *Derivation of the model*

Suppose that there are N_A superblocks, originally labelled $\nu = 1,2,\ldots,N_A$, and that superblock ν contains B_ν blocks, which are originally labelled $\xi(\nu) = 1,2,\ldots,B_\nu$. Further, suppose that block $\xi(\nu)$ contains $K_{\xi(\nu)}$ units (plots), which are originally labelled $\pi[\xi(\nu)] = 1,2,\ldots,K_{\xi(\nu)}$. The randomization of superblocks can then be understood as choosing at random a permutation of numbers $1,2,\ldots,N_A$, and then renumbering the superblocks with $h = 1,2,\ldots,N_A$ according to the positions of their original labels taken in the random permutation. Similarly, the randomization of blocks within superblock ν consists in selecting at random a permutation of numbers $1,2,\ldots,B_\nu$, and then renumbering the blocks with $j = 1,2,\ldots,B_\nu$ accordingly. Finally, the randomization of units within block $\xi(\nu)$ can be seen as selecting at random a permutation of numbers $1,2,\ldots,K_{\xi(\nu)}$, and then renumbering the units of the block with $l = 1,2,\ldots,K_{\xi(\nu)}$ accordingly (see Caliński and Kageyama, 1991, p. 99). The usual assumption will be made that any permutation of superblock labels can be selected with equal probability, that any permutation of block labels within a superblock can be selected with equal probability, as well as that any permutation of unit labels within a block can be selected in that way. Finally, it will be assumed that the randomizations of units within the blocks are among the blocks independent, that the randomizations of blocks within the superblocks are among the superblocks independent and also independent of the randomizations of units, and that all these randomizations are independent of the randomization of superblocks.

Utilizing the concept of a "null" experiment (cf. Nelder, 1965), let the response of the unit labelled $\pi[\xi(\nu)]$ be denoted by $\mu_{\pi[\xi(\nu)]}$, and let it be denoted by $m_{l[j(h)]}$ if in result of the randomizations the superblock originally labelled ν receives label h, the block originally labelled $\xi(\nu)$ receives label j and the unit originally labelled $\pi[\xi(\nu)]$ receives label l. Now, introducing the linear identity

$$\mu_{[\xi(\nu)]} = \mu_{\cdot[\cdot(\cdot)]} + (\mu_{\cdot[\cdot(\nu)]} - \mu_{\cdot[\cdot(\cdot)]}) + (\mu_{\cdot[\xi(\nu)]} - \mu_{\cdot[\cdot(\nu)]}) + (\mu_{\pi[\xi(\nu)]} - \mu_{\cdot[\xi(\nu)]}) \quad ,$$

where (according to the usual dot notation)

$$\mu_{\cdot[\xi(\nu)]} = K_{\xi(\nu)}^{-1} \sum_{\pi[\xi(\nu)]=1}^{K_{\xi(\nu)}} \mu_{\pi[\xi(\nu)]} \, , \quad \mu_{\cdot[\cdot(\nu)]} = B_{\nu}^{-1} \sum_{\xi(\nu)=1}^{B_{\nu}} \mu_{\cdot[\xi(\nu)]}$$

and

$$\mu_{\cdot[\cdot(\cdot)]} = N_A^{-1} \sum_{\nu=1}^{N_A} \mu_{\cdot[\cdot(\nu)]} \, ,$$

and defining the variance components (following Nelder, 1977)

$$\sigma_A^2 = (N_A - 1)^{-1} \sum_{\nu=1}^{N_A} (\mu_{\cdot[\cdot(\nu)]} - \mu_{\cdot[\cdot(\cdot)]})^2 \, ,$$

$$\sigma_B^2 = N_A^{-1} \sum_{\nu=1}^{N_A} \sigma_{B,\nu}^2 \, ,$$

where

$$\sigma_{B,\nu}^2 = (B_{\nu} - 1)^{-1} \sum_{\xi(\nu)=1}^{B_{\nu}} (\mu_{\cdot[\xi(\nu)]} - \mu_{\cdot[\cdot(\nu)]})^2 \, ,$$

and

$$\sigma_U^2 = N_A^{-1} \sum_{\nu=1}^{N_A} B_{\nu}^{-1} \sum_{\xi(\nu)=1}^{B_{\nu}} \sigma_{U,\xi(\nu)}^2 \, ,$$

where

$$\sigma_{U,\xi(\nu)}^2 = (K_{\xi(\nu)} - 1)^{-1} \sum_{\pi[\xi(\nu)]=1}^{K_{\xi(\nu)}} (\mu_{\pi[\xi(\nu)]} - \mu_{\cdot[\xi(\nu)]})^2 \, ,$$

and also introducing the weighted harmonic averages B_H and K_H defined as

$$B_H^{-1} = N_A^{-1} \sum_{\nu=1}^{N_A} B_\nu^{-1} \sigma_{B,\nu}^2 / \sigma_B^2$$

and

$$K_H^{-1} = N_A^{-1} \sum_{\nu=1}^{N_A} B_\nu^{-1} \sum_{\xi(\nu)=1}^{B_\nu} K_{\xi(\nu)}^{-1} \sigma_{U,\xi(\nu)}^2 / \sigma_U^2 ,$$

accordingly, one can write the linear model

$$m_{l[j(h)]} = \mu + \alpha_h + \beta_{j(h)} + \eta_{l[j(h)]} , \tag{2.1}$$

for any indices h, j and l resulting from the randomizations, where $\mu = \mu_{\cdot[\cdot(\cdot)]}$ is a constant parameter, while α_h, $\beta_{j(h)}$ and $\eta_{l[j(h)]}$ are random variables, the first representing a superblock random effect, the second representing a block random effect and the third a unit error. The following moments of these random variables are easily obtainable:

$$\mathrm{E}(\alpha_h) = \mathrm{E}(\beta_{j(h)}) = \mathrm{E}(\eta_{l[j(h)]}) = 0 ,$$

$$\mathrm{Cov}(\alpha_h, \beta_{j(h')}) = \mathrm{Cov}(\alpha_h, \eta_{l[j(h')]}) = 0 , \text{ whether } h = h' \text{ or } h \neq h' ,$$

$$\mathrm{Cov}(\beta_{j(h)}, \eta_{l[j'(h')]}) = 0 , \text{ whether } j(h) = j'(h') \text{ or } j(h) \neq j'(h') ,$$

$$\mathrm{Cov}(\alpha_h, \alpha_{h'}) = \begin{cases} N_A^{-1}(N_A - 1)\sigma_A^2 & \text{if } h = h' , \\ -N_A^{-1}\sigma_A^2 & \text{if } h \neq h' , \end{cases}$$

$$\mathrm{Cov}(\beta_{j(h)}, \beta_{j'(h')}) = \begin{cases} B_H^{-1}(B_H - 1)\sigma_B^2 & \text{if } h = h' \text{ and } j = j' , \\ -B_H^{-1}\sigma_B^2 & \text{if } h = h' \text{ and } j \neq j' , \\ 0 & \text{if } h \neq h' , \end{cases}$$

and

$$\mathrm{Cov}(\eta_{l[j(h)]}, \eta_{l'[j'(h')]}) = \begin{cases} K_H^{-1}(K_H - 1)\sigma_U^2 & \text{if } h = h' , j = j' , l = l' , \\ -K_H^{-1}\sigma_U^2 & \text{if } h = h' , j = j' , l \neq l' , \\ 0 & \text{if } j(h) \neq j'(h') . \end{cases}$$

(The derivations are straightforward, similar to those used in Caliński and Kageyama, 1988.)

Thus, the responses $\{m_{l[j(h)]}\}$ have in the conceptual null experiment the model (2.1) with the properties

$$\mathrm{E}(m_{l[j(h)]}) = \mu$$

and

$$\mathrm{Cov}(m_{l[j(h)]}, m_{l'[j'(h')]}) = (\delta_{hh'} - N_A^{-1})\,\sigma_A^2 + \delta_{hh'}(\delta_{jj'} - B_H^{-1})\,\sigma_B^2 + \delta_{hh'}\delta_{jj'}(\delta_{ll'} - K_H^{-1})\,\sigma_U^2\ ,$$

where the δ's are the usual Kronecker deltas, denoting 1 if the indices coincide, and 0 otherwise.

Further, taking into account (as in Caliński and Kageyama, 1991, p. 101) the technical error and denoting it by $e_{l[j(h)]}$, the model of the variable observed on unit $l\,[\,j(h)\,]$ in the null experiment can be written as

$$y_{l\,[\,j(h)\,]} = m_{l\,[\,j(h)\,]} + e_{l\,[\,j(h)\,]} = \mu + \alpha_h + \beta_j + \eta_{l\,[\,j(h)\,]} + e_{l\,[\,j(h)\,]}\ , \tag{2.2}$$

for any h, j and l. As usual, it will be assumed that the technical errors $\{e_{l\,[\,j(h)\,]}\}$ are uncorrelated, with zero expectation and constant variance, σ_e^2, and that they are independent of the remaining random variables in the model. Hence, the first and second moments of the random variables $\{y_{l[\,j(h)]}\}$ defined in (2.2) have the forms

$$\mathrm{E}(y_{l[\,j(h)]}) = \mu$$

and

$$\mathrm{Cov}(y_{l[\,j(h)]}\,,\ y_{l'[\,j'(h')]}) = (\delta_{hh'} - N_A^{-1})\,\sigma_A^2 + \delta_{hh'}(\delta_{jj'} - B_H^{-1})\,\sigma_B^2$$

$$+\ \delta_{hh'}\delta_{jj'}(\delta_{ll'} - K_H^{-1})\,\sigma_U^2 + \delta_{hh'}\,\delta_{jj'}\,\delta_{ll'}\,\sigma_e^2\ , \tag{2.3}$$

for all $l[\,j(h)]$ and all $l'[\,j'(h')]$.

Similarly as in the model considered by Caliński and Kageyama (1991, p. 101), it follows from these moments that the superblocks, the blocks within superblocks and the units within the blocks, can be regarded as "homogenous" in the sense that the observed responses of the units may, under the same treatment, be considered as observations on random variables $\{\,y_{l[\,j(h)]}\}$ exchangeable, without affecting their moments, individually within a block, in sets among the blocks within a superblock, as far as the sizes of the blocks allow for this, and also in sets of such sets among the superblocks, as far as the block sizes and the numbers of blocks within the superblocks allow for this.

With regard to this type of homogeneity of units, blocks and superblocks, the randomization principal can be obeyed in designing an experiment according to a chosen incidence matrix $\mathbf{N} = [\mathbf{N}_1 : \mathbf{N}_2 : \ldots : \mathbf{N}_a]$ by adopting the following rule. The a submatrices $\mathbf{N}_h$, $h = 1,2,..,a$, are assigned to a out of the N_A available superblocks by assigning the h-th submatrix to that superblock which due to the randomization has label h, and the b_h columns of $\mathbf{N}_h$ are assigned to b_h out of the B_h blocks available in the h-th superblock by assigning the $j(h)$-th column of

and

$$\mathrm{Cov}(m_{l[j(h)]}, m_{l'[j'(h')]}) = (\delta_{hh'} - N_A^{-1})\,\sigma_A^2 + \delta_{hh'}(\delta_{jj'} - B_H^{-1})\,\sigma_B^2 + \delta_{hh'}\delta_{jj'}(\delta_{ll'} - K_H^{-1})\,\sigma_U^2 ,$$

where the δ's are the usual Kronecker deltas, denoting 1 if the indices coincide, and 0 otherwise.

Further, taking into account (as in Caliński and Kageyama, 1991, p. 101) the technical error and denoting it by $e_{l[j(h)]}$, the model of the variable observed on unit $l\,[j(h)]$ in the null experiment can be written as

$$y_{l[j(h)]} = m_{l[j(h)]} + e_{l[j(h)]} = \mu + \alpha_h + \beta_j + \eta_{l[j(h)]} + e_{l[j(h)]} , \tag{2.2}$$

for any h, j and l. As usual, it will be assumed that the technical errors $\{e_{l[j(h)]}\}$ are uncorrelated, with zero expectation and constant variance, σ_e^2, and that they are independent of the remaining random variables in the model. Hence, the first and second moments of the random variables $\{y_{l[j(h)]}\}$ defined in (2.2) have the forms

$$\mathrm{E}(y_{l[j(h)]}) = \mu$$

and

$$\mathrm{Cov}(y_{l[j(h)]}, y_{l'[j'(h')]}) = (\delta_{hh'} - N_A^{-1})\,\sigma_A^2 + \delta_{hh'}(\delta_{jj'} - B_H^{-1})\,\sigma_B^2$$
$$+ \delta_{hh'}\delta_{jj'}(\delta_{ll'} - K_H^{-1})\,\sigma_U^2 + \delta_{hh'}\delta_{jj'}\delta_{ll'}\sigma_e^2 , \tag{2.3}$$

for all $l[j(h)]$ and all $l'[j'(h')]$.

Similarly as in the model considered by Caliński and Kageyama (1991, p. 101), it follows from these moments that the superblocks, the blocks within superblocks and the units within the blocks, can be regarded as "homogenous" in the sense that the observed responses of the units may, under the same treatment, be considered as observations on random variables $\{y_{l[j(h)]}\}$ exchangeable, without affecting their moments, individually within a block, in sets among the blocks within a superblock, as far as the sizes of the blocks allow for this, and also in sets of such sets among the superblocks, as far as the block sizes and the numbers of blocks within the superblocks allow for this.

With regard to this type of homogeneity of units, blocks and superblocks, the randomization principal can be obeyed in designing an experiment according to a chosen incidence matrix $\mathbf{N} = [\mathbf{N}_1 : \mathbf{N}_2 : \ldots : \mathbf{N}_a]$ by adopting the following rule. The a submatrices $\mathbf{N}_h$, $h = 1,2,\ldots,a$, are assigned to a out of the N_A available superblocks by assigning the h-th submatrix to that superblock which due to the randomization has label h, and the b_h columns of $\mathbf{N}_h$ are assigned to b_h out of the B_h blocks available in the h-th superblock by assigning the $j(h)$-th column of

$\mathbf{N}_h$ to that block which due to the randomization has label $j(h)$. This is to be accomplished for $h = 1,2,\ldots,a$. Then the treatments indicated by the $j(h)$-th column of $\mathbf{N}$ are assigned to the experimental units of the block labelled $j(h)$, in numbers defined by the corresponding elements of $\mathbf{N}_h = [n_{ij(h)}]$, i.e. the i-th treatment to $n_{ij(h)}$ units, in the order determined by the labels the units of the block have received due to the randomization. This rule implies not only that $a \leq N_A$, but also that $b_h \leq B_h$ for $h = 1,2,\ldots,a$, and that the units in the available blocks are in sufficient numbers with regard to the vector of block sizes $\mathbf{k}$, i.e. that $\max_{j(h)} (k_{j(h)}) \leq \min_{\xi(\nu)} K_{\xi(\nu)}$. This means that either the choice of $\mathbf{N}$ is to be conditioned by the above constraint, or an adjustment of $\mathbf{N}$ is to be made after the randomization of blocks and superblocks (as suggested by White, 1975, p. 558).

Now, adopting the assumption of complete additivity, as in Caliński and Kageyama (1991, p. 102), equivalent to the assumption that the variances and covariances of the variables $\{\alpha_h\}$, $\{\beta_j\}$, $\{\eta_{l[j(h)]}\}$ and $\{e_{l[j(h)]}\}$ do not depend on the treatment applied, the adjustment of the model (2.2) to a real experimental situation of comparing several treatments, v, in the same experiment can be made by changing the constant term only. Thus, the model gets the form

$$y_{l[j(h)]}(i) = \mu(i) + \alpha_h + \beta_{j(h)} + \eta_{l[j(h)]} + e_{l[j(h)]}\ , \tag{2.4}$$

for $i = 1,2,\ldots,v$, $h = 1,2,\ldots,a$, $j(h) = 1,2,\ldots,b_h$, $l[j(h)] = 1,2,\ldots,k_{j(h)}$, with

$$\mathrm{E}[y_{l[j(h)]}(i)] = \mu(i) = N_A^{-1} \sum_{\nu=1}^{N_A} B_\nu^{-1} \sum_{\xi(\nu)=1}^{B_{(\nu)}} K_{\xi(\nu)}^{-1} \sum_{\pi[\xi(\nu)]=1}^{K_{\xi(\nu)}} \mu_{\pi[\xi(\nu)]}(i)\ , \tag{2.5}$$

where $\mu_{\pi[\xi(\nu)]}(i)$ is the true response of unit π in block ξ within superblock ν to treatment i, and with

$$\mathrm{Cov}[y_{l[j(h)]}(i),\ y_{l'[j'(h')]}(i')] = \mathrm{Cov}(y_{l[j(h)]},\ y_{l'[j'(h')]}), \tag{2.6}$$

as given explicitly in (2.3).

Finally, writing the observed variables $\{y_{l[j(h)]}(i)\}$ in form of an $n \times 1$ vector $\mathbf{y} = [\mathbf{y}'_1, \mathbf{y}'_2, \ldots, \mathbf{y}'_a]'$, where the subvector $\mathbf{y}_h$ represents the variables observed on the $n_h = \Sigma_{j(h)=1}^{b_h} k_{j(h)}$ units of the superblock h, and the corresponding unit error and technical error variables in form of $n \times 1$ vectors $\boldsymbol{\eta}$ and $\mathbf{e}$, respectively, and also writing the treatment parameters as $\boldsymbol{\tau} = [\tau_1, \tau_2, \ldots, \tau_v]'$, where $\tau_i = \mu(i)$, the superblock variables as $\boldsymbol{\alpha} = [\alpha_1, \alpha_2, \ldots, \alpha_a]'$ and the block variables as $\boldsymbol{\beta} = [\boldsymbol{\beta}'_1, \boldsymbol{\beta}'_2, \ldots, \boldsymbol{\beta}'_a]'$, where $\boldsymbol{\beta}_h = [\beta_{1(h)}, \beta_{2(h)}, \ldots, \beta_{b_h(h)}]'$, one can express the model (2.4) in matrix notation as

$$\mathbf{y} = \boldsymbol{\Delta}'\boldsymbol{\tau} + \mathbf{G}'\boldsymbol{\alpha} + \mathbf{D}'\boldsymbol{\beta} + \boldsymbol{\eta} + \mathbf{e}\ , \tag{2.7}$$

and the corresponding moments (2.5) and (2.6) in form of the expectation vector

$$\mathrm{E}(\mathbf{y}) = \mathbf{\Delta}'\boldsymbol{\tau} \tag{2.8}$$

and the dispersion matrix (covariance matrix)

$$\mathrm{Cov}(\mathbf{y}) = (\mathbf{G}'\mathbf{G} - N_A^{-1}\mathbf{1}_n\mathbf{1}'_n)\,\sigma_A^2 + (\mathbf{D}'\mathbf{D} - B_H^{-1}\mathbf{G}'\mathbf{G})\,\sigma_B^2 + (\mathbf{I}_n - K_H^{-1}\mathbf{D}'\mathbf{D})\,\sigma_U^2 + \mathbf{I}_n\sigma_e^2\,, \tag{2.9}$$

respectively (see Section 1 for definitions of the matrices involved).

Note that the model (2.7), with properties (2.8) and (2.9), can be seen as a generalization of the model used by Patterson and Thompson (1971), when writing the dispersion matrix (2.9) in the form

$$\mathrm{Cov}(\mathbf{y}) = \sigma_1^2(\mathbf{G}'\mathbf{\Gamma}_1\mathbf{G} + \mathbf{D}'\mathbf{\Gamma}_2\mathbf{D} + \mathbf{I}_n)\,,$$

where

$$\mathbf{\Gamma}_1 = \mathbf{I}_a\gamma_1 - N_A^{-1}\mathbf{1}_a\mathbf{1}'_a\sigma_A^2/\sigma_1^2, \quad \gamma_1 = (\sigma_A^2 - B_H^{-1}\sigma_B^2)/\sigma_1^2, \quad \sigma_1^2 = \sigma_U^2 + \sigma_e^2,$$

and

$$\mathbf{\Gamma}_2 = \mathbf{I}_b\gamma_2, \quad \gamma_2 = (\sigma_B^2 - K_H^{-1}\sigma_U^2)/\sigma_1^2\,.$$

Furthermore, note that if

$$B_1 = B_2 = \ldots = B_{N_A} = B \text{ (say)} \quad \text{and} \quad K_{1(\nu)} = K_{2(\nu)} = \ldots = K_{B(\nu)} = K \text{ (say)}$$

for each ν (=1,2,...,N_A), then the present model is comparable with that recently considered by Mejza (1992, p. 269).

2.2. Main estimation results

Under the present model the following main results concerning the linear estimation of treatment parametric functions are obtainable. These results are relevant generalizations of those given in Section 2.2 of Caliński and Kageyama (1991).

Theorem 2.1. Under the model (2.7), with properties (2.8) and (2.9), a function $\mathbf{w}'\mathbf{y}$ is uniformly the best linear unbiased estimator (BLUE) of $\mathbf{c}'\boldsymbol{\tau}$ if and only if $\mathbf{w} = \mathbf{\Delta}'\mathbf{s}$, where $\mathbf{s} = \mathbf{r}^{-\delta}\mathbf{c}$ satisfies the conditions

$$(\mathbf{k}^\delta - \mathbf{N}'\mathbf{r}^{-\delta}\mathbf{N})\mathbf{N}'\mathbf{s} = \mathbf{0} \tag{2.10}$$

and

$$(\mathbf{n}^\delta - \mathbf{R}'\mathbf{r}^{-\delta}\mathbf{R})\mathbf{R}'\mathbf{s} = \mathbf{0}\,. \tag{2.11}$$

Proof. On account of Theorem 3 of Zyskind (1967), a function $\mathbf{w}'\mathbf{y}$ is, under the considered model, the BLUE of its expectation, i.e. of $\mathbf{w}'\boldsymbol{\Delta}'\boldsymbol{\tau}$, if and only if

$$(\mathbf{I}_n - \mathbf{P}_{\Delta'})[(\mathbf{G}'\mathbf{G} - N_A^{-1}\mathbf{1}_n\mathbf{1}_n')\sigma_A^2 + (\mathbf{D}'\mathbf{D} - B_H^{-1}\mathbf{G}'\mathbf{G})\sigma_B^2$$
$$+ (\mathbf{I}_n - K_H^{-1}\mathbf{D}'\mathbf{D})\sigma_U^2 + \mathbf{I}_n\sigma_e^2]\mathbf{w} = \mathbf{0},$$

where $\mathbf{P}_{\Delta'} = \boldsymbol{\Delta}'\mathbf{r}^{-\delta}\boldsymbol{\Delta}$ denotes the orthogonal projector on $C(\boldsymbol{\Delta}')$, the column space of $\boldsymbol{\Delta}'$. If this is to hold uniformly for any set of the variance components σ_A^2, σ_B^2, σ_U^2 and σ_e^2, it is necessary and sufficient that $(\mathbf{I}_n - \mathbf{P}_{\Delta'})\mathbf{w} = \mathbf{0}$, $(\mathbf{I}_n - \mathbf{P}_{\Delta'})\mathbf{D}'\mathbf{D}\mathbf{w} = \mathbf{0}$ and $(\mathbf{I}_n - \mathbf{P}_{\Delta'})\mathbf{G}'\mathbf{G}\mathbf{w} = \mathbf{0}$. But these equations hold simultaneously if and only if $\mathbf{w} = \boldsymbol{\Delta}'\mathbf{s}$, $(\mathbf{I}_n - \mathbf{P}_{\Delta'})\mathbf{D}'\mathbf{D}\boldsymbol{\Delta}'\mathbf{s} = \mathbf{0}$ and $(\mathbf{I}_n - \mathbf{P}_{\Delta'})\mathbf{G}'\mathbf{G}\boldsymbol{\Delta}'\mathbf{s} = \mathbf{0}$ for some $\mathbf{s}$. However, the latter two equations are equivalent to (2.10) and (2.11), respectively. □

In connection with this proof note that the component $N_B^{-1}\mathbf{1}_n\mathbf{1}_n'\sigma_A^2$ in (2.9) does not play any role in establishing Theorem 2.1, since $(\mathbf{I}_n - \mathbf{P}_{\Delta'})\mathbf{1}_n = \mathbf{0}$. For the same reason the simplification of $\boldsymbol{\Gamma}_1$ to the form $\mathbf{I}_n\gamma_1$, as suggested by Patterson and Thompson (1971), does not affect the BLUE of $\mathbf{c}'\boldsymbol{\tau}$.

Corollary 2.1. For the estimation of $\mathbf{c}'\boldsymbol{\tau} = \mathbf{s}'\mathbf{r}^{\delta}\boldsymbol{\tau}$ under the model as in Theorem 2.1, the following applies.

(a) If $\mathbf{N}'\mathbf{s} = 0$, which implies $\mathbf{R}'\mathbf{s} = \mathbf{0}$, the conditions (2.10) and (2.11) are satisfied, and the estimated function is a contrast, i.e., $\mathbf{c}'\mathbf{1}_v = \mathbf{s}'\mathbf{r} = 0$.

(b) If $\mathbf{N}'\mathbf{s} \neq \mathbf{0}$, but $\mathbf{R}'\mathbf{s} = \mathbf{0}$, then (2.11) is satisfied, but to satisfy (2.10) it is necessary and sufficient that the elements of $\mathbf{N}'\mathbf{s}$ obtained from the same connected subdesign of D^* are all equal.

(c) If $\mathbf{R}'\mathbf{s} \neq \mathbf{0}$, which implies that $\mathbf{N}'\mathbf{s} \neq \mathbf{0}$, then to satisfy (2.11) in addition to (2.10) it is necessary and sufficient that not only the elements of $\mathbf{N}'\mathbf{s}$ from the same connected subdesign of D^* are all equal, but also the elements of $\mathbf{R}'\mathbf{s}$ obtained from the same connected subdesign of D are all equal.

Proof. It follows exactly the same pattern as the proof of Corollary 2.1 in Caliński and Kageyama (1991). □

Now a question arises, under which design conditions any function $\mathbf{s}'\boldsymbol{\Delta}\mathbf{y}$ is the BLUE of its expectation. An answer to this can be given as follows.

Theorem 2.2. Under the model as in Theorem 2.1, any function $\mathbf{w}'\mathbf{y} = \mathbf{s}'\boldsymbol{\Delta}\mathbf{y}$, i.e. with any $\mathbf{s}$, is uniformly the BLUE of $\mathrm{E}(\mathbf{w}'\mathbf{y}) = \mathbf{s}'\mathbf{r}^{\delta}\boldsymbol{\tau}$ if and only if

(i) both of the designs, D^* and D, are orthogonal (in the sense recalled by Caliński, 1993, Definition 2.7),

(ii) the block sizes are constant within any connected subdesign of D^* and the sizes of superblocks are constant within any connected subdesign of D.

Proof. From the proof of Theorem 2.1 it is evident that $\mathbf{s}'\mathbf{\Delta y}$ is the BLUE of its expectation for any $\mathbf{s}$ if and only if

$$(\mathbf{I}_n - \mathbf{P}_{\Delta'})\mathbf{D}'\mathbf{D}\mathbf{\Delta}' = \mathbf{0} \tag{2.12}$$

and

$$(\mathbf{I}_n - \mathbf{P}_{\Delta'})\mathbf{G}'\mathbf{G}\mathbf{\Delta}' = \mathbf{0}\ . \tag{2.13}$$

Adopting the same reasoning as that used in the proof of Theorem 2.2 of Caliński and Kageyama (1991), the present results can be proved. □

Note that Remark 2.1 of Caliński and Kageyama (1991) applies here as well, with an obvious extension to the design D. In particular, if the latter is connected, which usually is the case, then the orthogonality condition for it can be written as $\mathbf{R} = n^{-1}\mathbf{r}\mathbf{n}'$, which under the additional condition (ii) reduces to $\mathbf{R} = (1/a)\mathbf{r}\mathbf{1}'_a$.

Remark 2.1. If the condition (i) and (ii) stated in Theorem 2.2 are satisfied, i.e. if (2.12) and (2.13) hold, then

$$\mathrm{Cov}(\mathbf{y})\mathbf{\Delta}'$$
$$= \mathbf{\Delta}'\mathbf{r}^{-\delta}[(\mathbf{RR}' - N_A^{-1}\mathbf{rr}')\sigma_A^2 + (\mathbf{NN}' - B_H^{-1}\mathbf{RR}')\sigma_B^2 + (\mathbf{r}^\delta - K_H^{-1}\mathbf{NN}')\sigma_U^2 + \mathbf{r}^\delta\sigma_e^2]\ ,$$

which implies that both $\mathbf{\Delta}'\mathbf{s}$ and $\mathrm{Cov}(\mathbf{y})\mathbf{\Delta}'\mathbf{s}$ belong to $C(\mathbf{\Delta}')$ for any $\mathbf{s}$, and thus, by Theorem 4 of Zyskind (1967), the BLUEs obtainable under the model (2.7), with moments (2.8) and (2.9), can equivalently be obtained under a simple alternative model in which the dispersion matrix (2.9) is reduced to $\mathbf{I}_n$ multiplied by a positive scalar (see also Rao and Mitra, 1971, Section 8.2). Moreover, it can be shown (applying, e.g., Theorem 2.3.2 of Rao and Mitra, 1971) that the equalities (2.12) and (2.13) are not only sufficient but also necessary conditions for the BLUEs obtainable under the two alternative models to be the same. Thus, (2.12) and (2.13) are necessary and sufficient for $\mathbf{s}'\mathbf{\Delta y}$ to be both the simple least squares estimator (SLSE) and the BLUE of its expectation, $\mathbf{s}'\mathbf{r}^\delta\boldsymbol{\tau}$, whichever vector $\mathbf{s}$ is used.

3. Resolving into stratum submodels

Results of Section 2 are rather discouraging, similarly as those of section 2 in Caliński and Kageyama (1991). According to the results obtained there, in many NB designs the BLUEs will exist under the model (2.7) for exceptional parametric functions of interest only, or for none of them.

This difficulty with the model (2.7) can be evaded by resolving it into four submodels (one more than in the case of an ordinary block design; see Caliński and Kageyama, 1991, p. 105), in accordance with the stratification of the experimental units. In fact, the units of a NB design can be seen as being grouped according to a nested classification with four strata. The strata may be defined as follows:

1st stratum – of units within blocks, called "intra-block",

2nd stratum – of blocks within superblocks, called "inter-block - intra-superblock",

3rd stratum – of superblocks within the experimental area, called "inter-superblock",

4th stratum – of the total experimental area.

Using Nelder's (1965) notation, this "block-structure" can be represented by the relation

Units (plots) → Blocks → Superblocks → Total area.

Due to this stratification, the observed vector **y** can be decomposed as

$$\mathbf{y} = \mathbf{y}_1 + \mathbf{y}_2 + \mathbf{y}_3 + \mathbf{y}_4 , \tag{3.1}$$

where each of the four components is related to one of the strata. The component vectors $\mathbf{y}_f$, $f = 1,2,3,4$, are thus obtainable by projecting **y** orthogonally on relevant subspaces, mutually orthogonal. The first component in (3.1) can be written as

$$\mathbf{y}_1 = \boldsymbol{\varphi}_1\mathbf{y} , \tag{3.2}$$

where $\boldsymbol{\varphi}_1 = \mathbf{I}_n - \mathbf{D}'\mathbf{k}^{-\delta}\mathbf{D} = \mathbf{I}_n - \mathbf{P}_{\mathbf{D}'}$ is exactly as defined in (3.3) of Caliński and Kageyama (1991), i.e., $\mathbf{y}_1$ is the orthogonal projection of **y** on $C^{\perp}(\mathbf{D}')$, the orthogonal complement of $C(\mathbf{D}')$. The second component is

$$\mathbf{y}_2 = \boldsymbol{\varphi}_2\mathbf{y} , \tag{3.3}$$

where now $\boldsymbol{\varphi}_2 = \mathbf{D}'\mathbf{k}^{-\delta}\mathbf{D} - \mathbf{G}'\mathbf{n}^{-\delta}\mathbf{G} = \mathbf{P}_{\mathbf{D}'} - \mathbf{P}_{\mathbf{G}'}$, with $\mathbf{n}^{-\delta} = (\mathbf{n}^{\delta})^{-1} = (\mathbf{G}\mathbf{G}')^{-1}$, i.e., $\mathbf{y}_2$ is the orthogonal projection of **y** on $C^{\perp}(\mathbf{G}') \cap C(\mathbf{D}')$, the orthogonal complement of $C(\mathbf{G}')$ in $C(\mathbf{D}')$. The third is

$$\mathbf{y}_3 = \boldsymbol{\varphi}_3\mathbf{y} , \tag{3.4}$$

where $\boldsymbol{\varphi}_3 = \mathbf{G}'\mathbf{n}^{-\delta}\mathbf{G} - n^{-1}\mathbf{1}_n\mathbf{1}'_n = \mathbf{P}_{\mathbf{G}'} - \mathbf{P}_{\mathbf{1}_n}$, i.e. $\mathbf{y}_3$ is the orthogonal projection of **y** on $C^{\perp}(\mathbf{1}_n) \cap C(\mathbf{G}')$, the orthogonal complement of $C(\mathbf{1}_n)$ in $C(\mathbf{G}')$. Finally, the fourth component is

$$\mathbf{y}_4 = \boldsymbol{\varphi}_4\mathbf{y} , \tag{3.5}$$

where $\boldsymbol{\varphi}_4 = n^{-1}\mathbf{1}_n\mathbf{1}'_n = \mathbf{P}_{\mathbf{1}_n}$, i.e., $\mathbf{y}_4$ is the orthogonal projection of **y** on $C(\mathbf{1}_n)$.

Evidently, the four matrices, φ_1, φ_2, φ_3 and φ_4 (comparable with those of Pearce, 1983, p. 153), satisfy the conditions

$$\varphi_f = \varphi_f', \ \varphi_f \varphi_f = \varphi_f, \ \varphi_f \varphi_{f'} = \mathbf{0} \quad \text{for} \ f \neq f', \ \text{where} \ f, f' = 1,2,3,4, \qquad (3.6)$$

and the condition $\varphi_1 + \varphi_2 + \varphi_3 + \varphi_4 = \mathbf{I}_n$, the third equality in (3.6) implying in particular that

$$\varphi_1 \mathbf{D}' = \mathbf{0}, \ \varphi_f \mathbf{G}' = \mathbf{0} \quad \text{for} \ f = 1,2, \quad \varphi_f \mathbf{1}_n = \mathbf{0} \quad \text{for} \ f = 1,2,3, \qquad (3.7)$$

while the first two equalities in (3.6) imply that

$$\text{rank}(\varphi_1) = n - b, \ \text{rank}(\varphi_2) = b - a, \ \text{rank}(\varphi_3) = a - 1 \ \text{and} \ \text{rank}(\varphi_4) = 1.$$

The projections (3.2), (3.3), (3.4) and (3.5) can be considered as submodels of the original overall model (2.7). They are of particular interest when the conditions (2.10) or/and (2.11) are not satisfied. Similarly as shown in Section 3 of Caliński and Kageyama (1991), the submodel (3.2) leads to the intra-block analysis, the submodel (3.3) leads to the inter-block-intra-superblock analysis, the submodel (3.4) to the inter-superblock analysis. The submodel (3.5) underlies the total-area or experimental-area analysis, suitable mainly for estimating the general parametric mean (hence the last stratum is sometimes called the "mean stratum"; see John, 1987, p. 184).

It will be now interesting to examine properties of the four submodels and their implications for the resulting analyses. Since there are close similarities to the results presented in Section 3 of Caliński and Kageyama (1991), only those results which are different will be indicated here. They will be presented without detailed proofs in most cases, as they would follow exactly the same patterns of reasoning that were used in the earlier paper.

3.1. The intra-block submodel

The submodel (3.2) has the properties

$$E(\mathbf{y}_1) = \varphi_1 \Delta' \boldsymbol{\tau} \ \text{and} \ \text{Cov}(\mathbf{y}_1) = \varphi_1(\sigma_U^2 + \sigma_e^2),$$

which are exactly the same as in (3.11) and (3.12) of Caliński and Kageyama (1991). Hence, the results concerning estimation of parameters and testing of relevant hypotheses are exactly as presented in Section 3.1 of that earlier paper.

In particular, any contrast $\mathbf{c}'\boldsymbol{\tau}$ such that $\mathbf{c} = \mathbf{C}_1\mathbf{s}$ for some $\mathbf{s}$, where $\mathbf{C}_1 = \Delta\varphi_1\Delta' = \mathbf{r}^\delta - \mathbf{N}\mathbf{k}^\delta\mathbf{N}'$, receives the BLUE under this submodel, in the form $\widehat{\mathbf{c}'\boldsymbol{\tau}} = \mathbf{s}'\Delta\mathbf{y}_1$. Its variance has the form

$$\text{Var}(\widehat{\mathbf{c}'\boldsymbol{\tau}}) = \mathbf{s}'\mathbf{C}_1\mathbf{s}(\sigma_U^2 + \sigma_e^2) = \mathbf{c}'\mathbf{C}_1^-\mathbf{c}(\sigma_U^2 + \sigma_e^2), \qquad (3.8)$$

where $\mathbf{C}_1^-$ is any generalized inverse (g-inverse) of $\mathbf{C}_1$.

The intra-block analysis of variance can be presented as

$$\mathbf{y}'\boldsymbol{\varphi}_1\mathbf{y} = \mathbf{Q}_1'\mathbf{C}_1^-\mathbf{Q}_1 + \mathbf{y}'\boldsymbol{\psi}_1\mathbf{y},$$

where $\mathbf{Q}_1\mathbf{C}_1^-\mathbf{Q}_1$, with $\mathbf{Q}_1 = \boldsymbol{\Delta}\boldsymbol{\varphi}_1\mathbf{y}$, is the intra-block treatment sum of squares, and $\mathbf{y}'\boldsymbol{\psi}_1\mathbf{y}$, with $\boldsymbol{\psi}_1 = \boldsymbol{\varphi}_1 - \boldsymbol{\varphi}_1\boldsymbol{\Delta}'\mathbf{C}_1^-\boldsymbol{\Delta}\boldsymbol{\varphi}_1 = \boldsymbol{\varphi}_1(\mathbf{I}_n - \boldsymbol{\Delta}'\mathbf{C}_1^-\boldsymbol{\Delta})\boldsymbol{\varphi}_1$, is the intra-block residual sum of squares, the first on $h_1 = \text{rank}(\mathbf{C}_1)$ degrees of freedom (d.f.), the second on $n - b - h_1 = \text{rank}(\boldsymbol{\psi}_1)$ d.f. The resulting intra-block residual mean square $s_1^2 = \mathbf{y}'\boldsymbol{\psi}_1\mathbf{y}/(n - b - h_1)$ is the minimum norm quadratic unbiased estimator (MINQUE) of $\sigma_1^2 = \sigma_U^2 + \sigma_e^2$, giving an unbiased estimator of (3.8), in the form

$$\hat{\text{Var}}(\mathbf{c}'\hat{\boldsymbol{\tau}}) = \mathbf{s}'\mathbf{C}_1\mathbf{s}\,s_1^2 = \mathbf{c}'\mathbf{C}_1^-\mathbf{c}\,s_1^2\,. \tag{3.9}$$

Under the multivariate normal distribution of $\mathbf{y}$, one can test the hypothesis $\boldsymbol{\tau}'\mathbf{C}_1\boldsymbol{\tau} = 0$, equivalent to $E(\mathbf{y}_1) = \mathbf{0}$ or $E(\mathbf{y}) \in \mathcal{C}(\mathbf{D}')$, by the variance ratio criterion $h_1^{-1}\mathbf{Q}_1'\mathbf{C}_1^-\mathbf{Q}_1/s_1^2$, which has then the F distribution with h_1 and $n - b - h_1$ d.f., central when the hypothesis is true.

3.2. The inter-block-intra-superblock submodel

The submodel (3.3), has the properties

$$E(\mathbf{y}_2) = \boldsymbol{\varphi}_2\boldsymbol{\Delta}'\boldsymbol{\tau} = \mathbf{D}'\mathbf{k}^{-\delta}\mathbf{D}\boldsymbol{\Delta}'\boldsymbol{\tau} - \mathbf{G}'\mathbf{n}^{-\delta}\mathbf{G}\,\boldsymbol{\Delta}'\boldsymbol{\tau}$$

and

$$\text{Cov}(\mathbf{y}_2) = \boldsymbol{\varphi}_2\mathbf{D}'\mathbf{D}\boldsymbol{\varphi}_2(\sigma_B^2 - K_H^{-1}\sigma_U^2) + \boldsymbol{\varphi}_2(\sigma_U^2 + \sigma_e^2),$$

similar to those of the inter-block submodel considered by Caliński and Kageyama (1991, Section 3.2), but now with different form of $\boldsymbol{\varphi}_2$.

The main estimation result under (3.3) can be expressed as follows.

Theorem 3.2. Under (3.2), a function $\mathbf{w}'\mathbf{y}_2 = \mathbf{w}'\boldsymbol{\varphi}_2\mathbf{y}$ is uniformly the BLUE of $\mathbf{c}'\boldsymbol{\tau}$ if and only if $\boldsymbol{\varphi}_2\mathbf{w} = \boldsymbol{\varphi}_2\boldsymbol{\Delta}'\mathbf{s}$, where the vectors $\mathbf{c}$ and $\mathbf{s}$ are in the relation $\mathbf{c} = \mathbf{C}_2\mathbf{s}$, $\mathbf{C}_2 = \boldsymbol{\Delta}\boldsymbol{\varphi}_2\boldsymbol{\Delta}'$, and in addition $\mathbf{s}$ satisfies the condition

$$[\tilde{\mathbf{K}}_0 - \tilde{\mathbf{N}}_0'(\tilde{\mathbf{N}}_0\mathbf{k}^{-\delta}\tilde{\mathbf{N}}_0')^-\tilde{\mathbf{N}}_0]\tilde{\mathbf{N}}_0'\mathbf{s} = \mathbf{0}\,, \tag{3.10}$$

or its equivalence

$$[\tilde{\mathbf{K}}_0 - \tilde{\mathbf{N}}_0\mathbf{r}^{-\delta}\tilde{\mathbf{N}}_0(\tilde{\mathbf{N}}_0\mathbf{r}^{-\delta}\tilde{\mathbf{N}}_0)^-\tilde{\mathbf{K}}_0]\tilde{\mathbf{N}}_0'\mathbf{s} = \mathbf{0}\,, \tag{3.11}$$

where

$$\tilde{\mathbf{K}}_0 = \text{diag}\,[\mathbf{K}_{01} : \mathbf{K}_{02} : \ldots : \mathbf{K}_{0a}\,],\quad \mathbf{K}_{0h} = \mathbf{k}_h^{\delta} - n_h^{-1}\mathbf{k}_h\mathbf{k}_h',$$

and

$$\tilde{\mathbf{N}}_0 = [\mathbf{N}_{01} : \mathbf{N}_{02} : \ldots : \mathbf{N}_{0a}], \quad \mathbf{N}_{0h} = \mathbf{N}_h - n_h^{-1}\mathbf{r}_h\mathbf{k}_h' .$$

Proof. A proof similar to that of Theorem 3.2 in Caliński and Kegeyama (1991) can be used. Here too, the following relations hold: $\mathbf{D}'\mathbf{k}^{-\delta}\mathbf{D}\boldsymbol{\varphi}_2 = \boldsymbol{\varphi}_2$, $\mathbf{D}\boldsymbol{\varphi}_2\mathbf{D}' = \tilde{\mathbf{K}}_0$, $\boldsymbol{\Delta}\boldsymbol{\varphi}_2\mathbf{D}' = \tilde{\mathbf{N}}_0$, $\mathbf{C}_2 = \boldsymbol{\Delta}\boldsymbol{\varphi}_2\boldsymbol{\Delta}' = \tilde{\mathbf{N}}_0\mathbf{k}^{-\delta}\tilde{\mathbf{N}}_0'$ and $\tilde{\mathbf{N}}_0\mathbf{k}^{-\delta}\tilde{\mathbf{K}}_0 = \tilde{\mathbf{N}}_0$. □

Corollary 3.1. For the estimation of $\mathbf{c}'\boldsymbol{\tau} = \mathbf{s}'\tilde{\mathbf{N}}_0\mathbf{k}^{-\delta}\tilde{\mathbf{N}}_0'\boldsymbol{\tau}$ under (3.2) the following applies:

(a) The case $\tilde{\mathbf{N}}_0'\mathbf{s} = \mathbf{0}$ is to be excluded.

(b) If $\tilde{\mathbf{N}}_0'\mathbf{s} \neq \mathbf{0}$, then $\mathbf{c}'\boldsymbol{\tau}$ is a contrast, and to satisfy (3.10) or (3.11) by the vector $\mathbf{s}$, it is necessary and sufficient that $\tilde{\mathbf{K}}_0\tilde{\mathbf{N}}_0'\mathbf{s} \in \mathcal{C}(\tilde{\mathbf{N}}_0'\mathbf{r}^{-\delta}\tilde{\mathbf{N}}_0) = \mathcal{C}(\tilde{\mathbf{N}}_0')$.

(c) If $\mathbf{s}$ is such that $\mathbf{r}_h'\mathbf{s} = \mathbf{0}$ for $h = 1,2,\ldots,a$, then the conditions (3.10) and (3.11) can be replaced by

$$\tilde{\mathbf{K}}_0\mathbf{N}'\mathbf{s} = \tilde{\mathbf{N}}_0'(\tilde{\mathbf{N}}_0\mathbf{k}^{-\delta}\tilde{\mathbf{N}}_0')^{-}\tilde{\mathbf{N}}_0\mathbf{N}'\mathbf{s}$$

and

$$\tilde{\mathbf{K}}_0\mathbf{N}'\mathbf{s} = \tilde{\mathbf{N}}_0'\mathbf{r}^{-\delta}\tilde{\mathbf{N}}_0(\tilde{\mathbf{N}}_0'\mathbf{r}^{-\delta}\tilde{\mathbf{N}}_0)^{-}\tilde{\mathbf{K}}_0\mathbf{N}'\mathbf{s} ,$$

respectively. To satisfy any of them it is then necessary and sufficient that $\tilde{\mathbf{K}}_0\mathbf{N}'\mathbf{s} \in \mathcal{C}(\tilde{\mathbf{N}}_0')$.

(d) If all block sizes are equal, i.e., $k_1 = k_2 = \ldots = k_b = k$ (say), then any of the conditions (3.10) and (3.11) is satisfied automatically by any $\mathbf{s}$.

Proof. The results (a), (b) and (c) can be proved exactly as in Corollary 3.1 of Caliński and Kageyama (1991). The result (d) is obtainable due to the equality $\tilde{\mathbf{K}}_0\tilde{\mathbf{N}}_0' = k\,\tilde{\mathbf{N}}_0$, where k is the constant block size, and Lemma 2.2.6(c) of Rao and Mitra(1971). □

Now it may be noted that if the conditions of Theorem 3.2 are satisfied, then any contrast $\mathbf{c}'\boldsymbol{\tau}$, such that $\mathbf{c} = \mathbf{C}_2\mathbf{s}$ for some $\mathbf{s}$, receives the BLUE under the submodel (3.2) in the form $\widehat{\mathbf{c}'\boldsymbol{\tau}} = \mathbf{s}'\boldsymbol{\Delta}\mathbf{y}_2 = \mathbf{c}'\mathbf{C}_2^{-}\boldsymbol{\Delta}\boldsymbol{\varphi}_2\mathbf{y}$. Its variance has the form

$$\mathrm{Var}(\widehat{\mathbf{c}'\boldsymbol{\tau}}) = \mathbf{s}'\tilde{\mathbf{N}}_0\tilde{\mathbf{N}}_0'\mathbf{s}(\sigma_B^2 - K_H^{-1}\sigma_U^2) + \mathbf{s}'\tilde{\mathbf{N}}_0\mathbf{k}^{-\delta}\tilde{\mathbf{N}}_0'\mathbf{s}(\sigma_U^2 + \sigma_e^2) . \quad (3.12)$$

Evidently, if $k_1 = k_2 = \ldots = k_b = k$, the variance (3.12) reduces to

$$\mathrm{Var}(\widehat{\mathbf{c}'\boldsymbol{\tau}}) = k^{-1}\mathbf{s}'\tilde{\mathbf{N}}_0\tilde{\mathbf{N}}_0'\mathbf{s}\,[k\sigma_B^2 + (1 - K_H^{-1}k)\,\sigma_U^2 + \sigma_e^2] =$$

$$\mathbf{c}'(k^{-1}\tilde{\mathbf{N}}_0\tilde{\mathbf{N}}_0')^{-}\mathbf{c}[k\sigma_B^2 + (1 - K_H^{-1}k)\,\sigma_U^2 + \sigma_e^2]. \quad (3.13)$$

An answer to the question what is necessary and sufficient for the condition (3.10), or (3.11), of Theorem 3.2 to be satisfied by any $\mathbf{s}$ is as follows.

Corollary 3.2. The condition (3.10) holds for any $\mathbf{s}$, i.e. the equality

$$\tilde{\mathbf{K}}_0\tilde{\mathbf{N}}'_0 = \tilde{\mathbf{N}}'_0(\tilde{\mathbf{N}}_0\mathbf{k}^{-\delta}\tilde{\mathbf{N}}'_0)^{-}\tilde{\mathbf{N}}_0\tilde{\mathbf{N}}'_0 \tag{3.14}$$

holds, if and only if for any vector $\mathbf{t}$ that satisfies the equality $\tilde{\mathbf{N}}_0\mathbf{t} = \mathbf{0}$ the equality $\tilde{\mathbf{N}}_0\tilde{\mathbf{K}}_0\mathbf{t} = \mathbf{0}$ holds too.

Proof. This can be proved exactly as Corollary 3.2 of Caliński and Kageyama (1991). □

Remark 3.1. Note that for $\tilde{\mathbf{N}}_0\mathbf{t} = \mathbf{0}$ to imply $\tilde{\mathbf{N}}_0\tilde{\mathbf{K}}_0\mathbf{t} = \mathbf{0}$ it is sufficient that $\mathbf{N}_{0h}\mathbf{t}_0 = 0$ implies $\mathbf{N}_{0h}\mathbf{K}_{0h}\mathbf{t}_h = 0$ for $h = 1,2,\dots,a$, with $\mathbf{t} = [\mathbf{t}'_1, \mathbf{t}'_2, \dots, \mathbf{t}'_a]'$. Moreover, for $\mathbf{N}_{0h}\mathbf{t}_h = \mathbf{0}$ to imply $\mathbf{N}_{0h}\mathbf{K}_{0h}\mathbf{t}_h = \mathbf{0}$ it is sufficient that $\mathbf{k}_h = k_h\mathbf{1}_{b_h}$, i.e. that the block sizes within the superblock h are all equal ($h = 1,2,\dots,a$).

Remark 3.2. (a) Since $\tilde{\mathbf{N}}_0\mathbf{1}_b = \mathbf{0}$ and $\tilde{\mathbf{N}}_0\tilde{\mathbf{K}}_0\mathbf{1}_b = \mathbf{0}$ always, the necessary and sufficient condition for the equality (3.14) of Corollary 3.2 can be replaced by the condition that $\mathbf{N}\mathbf{t}_0 = \mathbf{0}$ implies $\tilde{\mathbf{N}}_0\mathbf{k}^{\delta}\mathbf{t}_0 = \mathbf{0}$ for any vector $\mathbf{t}_0 = [\mathbf{t}'_{01}, \mathbf{t}'_{02}, \dots, \mathbf{t}'_{0a}]'$ such that $\mathbf{t}_{0h}$ is $\mathbf{k}_h^{\delta}$-orthogonal to $\mathbf{1}_{b_h}$ for any h, i.e. $\mathbf{k}'_h\mathbf{t}_{0h}$ for any h (= $1,2,\dots,a$).

(b) If rank $(\mathbf{N}_{0h}) = b_h - 1$ for any h, i.e., the columns of each $\mathbf{N}_h$ are linearly independent [as rank $(\mathbf{N}_{0h})$ = rank $(\mathbf{N}_h) - 1$], then a vector $\mathbf{t}_h$ that satisfies $\mathbf{N}_{0h}\mathbf{t}_h = \mathbf{0}$ must be equal or proportional to $\mathbf{1}_{b_h}$ [i.e., $\mathbf{t}_h \in C(\mathbf{1}_{b_h})$], and so satisfy also the equality $\mathbf{N}_{0h}\mathbf{K}_{0h}\mathbf{t}_h = \mathbf{0}$. Thus, the condition of Corollary 3.2 is then satisfied automatically, whatever the block sizes are.

Remark 3.3. If the condition (3.10) holds for any $\mathbf{s}$, i.e., if (3.14) holds, then

$$\mathrm{Cov}(\mathbf{y}_2)\boldsymbol{\varphi}_2\boldsymbol{\Delta}' = \boldsymbol{\varphi}_2\boldsymbol{\Delta}'\,[(\tilde{\mathbf{N}}_0\mathbf{k}^{-\delta}\tilde{\mathbf{N}}'_0)^{-}\tilde{\mathbf{N}}_0\tilde{\mathbf{N}}'_0(\sigma_B^2 - K_H^{-1}\sigma_U^2) + \mathbf{I}_n(\sigma_U^2 + \sigma_e^2)] \;,$$

which implies that both $\boldsymbol{\varphi}_2\boldsymbol{\Delta}'\mathbf{s}$ and $\mathrm{Cov}(\mathbf{y}_2)\boldsymbol{\varphi}_2\boldsymbol{\Delta}'\mathbf{s}$ belong to $C(\boldsymbol{\varphi}_2\boldsymbol{\Delta}')$ for any $\mathbf{s}$, and thus the condition stated in Theorem 4 of Zyskind (1967), when applied to Theorem 3.2, is satisfied. The implications of this are exactly as those indicated in Remark 3.5 of Caliński and Kageyama (1991). In particular it follows that (3.14) is necessary and sufficient for the BLUEs and SLSEs to coincide under this submodel.

Thus, if the equality (3.14) holds, then the inter-block-intra-superblock analysis of variance can be obtained, in the form

$$\mathbf{y}'\boldsymbol{\varphi}_2\mathbf{y} = \mathbf{Q}'_2\mathbf{C}_2^{-}\mathbf{Q}_2 + \mathbf{y}'\boldsymbol{\psi}_2\mathbf{y} \;,$$

where $\mathbf{Q}'_2\mathbf{C}_2^{-}\mathbf{Q}_2$, with $\mathbf{Q}_2 = \boldsymbol{\Delta}\boldsymbol{\varphi}_2\mathbf{y}$, is the inter-block (-intra-superblock) treatment sum of squares, and $\mathbf{y}'\boldsymbol{\psi}_2\mathbf{y}$, with $\boldsymbol{\psi}_2 = \boldsymbol{\varphi}_2 - \boldsymbol{\varphi}_2\boldsymbol{\Delta}'\mathbf{C}_2^{-}\boldsymbol{\Delta}\boldsymbol{\varphi}_2 = \boldsymbol{\varphi}_2(\mathbf{I}_n - \boldsymbol{\Delta}'\mathbf{C}_2^{-}\boldsymbol{\Delta})\boldsymbol{\varphi}_2$, is the inter-block (-intra-superblock) residual sum of squares, the first on $h_2 = \mathrm{rank}(\mathbf{C}_2) = \mathrm{rank}(\tilde{\mathbf{N}}_0)$ d.f., the second on $b - a - h_2 = \mathrm{rank}(\boldsymbol{\psi}_2)$ d.f. The resulting inter-block (-intra-superblock) residual mean square $s_2 = \mathbf{y}'\boldsymbol{\psi}_2\mathbf{y} / (b - a - h_2)$ is the MINQUE of

$$\sigma_2^2 = (b - a - h_2)^{-1}\{\mathrm{tr}(\tilde{\mathbf{K}}_0) - \mathrm{tr}[\tilde{\mathbf{N}}_0(\tilde{\mathbf{N}}_0\mathbf{k}^{-\delta}\tilde{\mathbf{N}}_0')^-\tilde{\mathbf{N}}_0]\}(\sigma_B^2 - K_H^{-1}\sigma_U^2) + \sigma_U^2 + \sigma_e^2 . \quad (3.15)$$

In the case of all k_j equal ($k_1 = k_2 = \ldots = k_b = k$), (3.15) is reduced to

$$\sigma_2^2 = k\sigma_B^2 + (1 - K_H^{-1}k)\,\sigma_U^2 + \sigma_e^2 , \quad (3.16)$$

further reducing to $\sigma_2^2 = k\sigma_B^2 + \sigma_e^2$ if $k = K_H$. It should be noted, however, that $b - a - h_2 = 0$ if $b - a = h_2$ (obviously $b - a \geq h_2$ always). In that case no estimator for σ_2^2 exists in the inter-block-intra-superblock analysis.

Thus, in the case of equal k_j's and $b - a > h_2$ the mean square s_2^2 can be used to obtain an unbiased estimator of (3.13), in the form

$$\hat{\mathrm{V}}\mathrm{ar}(\mathbf{c}'\hat{\boldsymbol{\tau}}) = k^{-1}\mathbf{s}'\tilde{\mathbf{N}}_0\tilde{\mathbf{N}}_0'\mathbf{s}\, s_2^2 = k\mathbf{c}'(\tilde{\mathbf{N}}_0\tilde{\mathbf{N}}_0')^-\mathbf{c}\, s_2^2 .$$

In general, the estimation of (3.12) is not so simple.

Furthermore, of $k_1 = k_2 = \ldots = k_b = k$, then $\mathrm{Cov}(\mathbf{y}_2) = \boldsymbol{\varphi}_2\sigma_2^2$, where σ_2^2 is as defined in (3.16), and under the multivariate normal distribution of $\mathbf{y}$ it is possible to test the hypothesis $\boldsymbol{\tau}'\mathbf{C}_2\boldsymbol{\tau} = 0$, equivalent to $\mathrm{E}(\mathbf{y}_2) = \mathbf{0}$ or $\mathbf{P}_{D'}\mathrm{E}(\mathbf{y}) \in \mathcal{C}(\mathbf{G}')$, by the variance ratio criterion $h_2^{-1}\mathbf{Q}_2'\mathbf{C}_2^{-1}\mathbf{Q}_2 / s_2^2$, which has then the F distribution with h_2 and $b - a - h_2$ d.f., central when the hypothesis is true. This, however, does not apply to the general case, when k_j's are not equal.

3.3. The inter-superblock submodel

The submodel (3.4) has the properties

$$\mathrm{E}(\mathbf{y}_3) = \boldsymbol{\varphi}_3\boldsymbol{\Delta}'\boldsymbol{\tau} = \mathbf{G}'\mathbf{n}^{-\delta}\mathbf{G}\boldsymbol{\Delta}'\boldsymbol{\tau} - n^{-1}\mathbf{1}_n\mathbf{r}'\boldsymbol{\tau} \quad (3.17)$$

and

$$\mathrm{Cov}(\mathbf{y}_3) = \boldsymbol{\varphi}_3\mathbf{G}'\mathbf{G}\boldsymbol{\varphi}_3(\sigma_A^2 - B_H^{-1}\sigma_B^2) + \boldsymbol{\varphi}_3\mathbf{D}'\mathbf{D}\,\boldsymbol{\varphi}_3(\sigma_B^2 - K_H^{-1}\sigma_U^2) + \boldsymbol{\varphi}_3(\sigma_U^2 + \sigma_e^2) . \quad (3.18)$$

The main estimation result under (3.4) can be expressed as follows

Theorem 3.3. Under (3.4), a function $\mathbf{w}'\mathbf{y}_3 = \mathbf{w}'\boldsymbol{\varphi}_3\mathbf{y}$ is uniformly the BLUE of $\mathbf{c}'\boldsymbol{\tau}$ if and only if $\boldsymbol{\varphi}_3\mathbf{w} = \boldsymbol{\varphi}_3\boldsymbol{\Delta}'\mathbf{s}$, where the vectors $\mathbf{c}$ and $\mathbf{s}$ are in the relation $\mathbf{c} = \mathbf{C}_3\mathbf{s}$, $\mathbf{C}_3 = \boldsymbol{\Delta}\boldsymbol{\varphi}_3\boldsymbol{\Delta}'$, and in addition $\mathbf{s}$ satisfies the conditions

$$\{\mathbf{K}_0 - \tilde{\mathbf{K}}_0 - (\mathbf{N}_0 - \tilde{\mathbf{N}}_0)'[(\mathbf{N}_0 - \tilde{\mathbf{N}}_0)\mathbf{k}^{-\delta}(\mathbf{N}_0 - \tilde{\mathbf{N}}_0)']^-(\mathbf{N}_0 - \tilde{\mathbf{N}}_0)\}(\mathbf{N}_0 - \tilde{\mathbf{N}}_0)'\mathbf{s} = \mathbf{0} \quad (3.19)$$

and

$$[\mathbf{L}_0 - \mathbf{R}'_0(\mathbf{R}_0\mathbf{n}^{-\delta}\mathbf{R}_0)^-\mathbf{R}_0']\mathbf{R}_0'\mathbf{s} = \mathbf{0} , \quad (3.20)$$

where $\mathbf{K}_0 = \mathbf{k}^{-\delta} - n^{-1}\mathbf{kk}'$ and $\mathbf{N}_0 = \mathbf{N} - n^{-1}\mathbf{rk}'$, as in Section 3.2 of Caliński and Kageyama (1991), while $\tilde{\mathbf{K}}_0$ and $\tilde{\mathbf{N}}_0$ are as in Section 3.2 of the present paper, $\mathbf{L}_0 = \mathbf{n}^{\delta} - n^{-1}\mathbf{nn}'$ and $\mathbf{R}_0 = \mathbf{R} - n^{-1}\mathbf{rn}'$, with $\mathbf{R}$ as defined in Section 1.

Proof. Under (3.4), with (3.17) and (3.18), the necessary and sufficient condition of Theorem 3 of Zyskind (1967) for a function $\mathbf{w}'\mathbf{y}_3 = \mathbf{w}'\boldsymbol{\varphi}_3\mathbf{y}$ to be the BLUE of $\mathrm{E}(\mathbf{w}'\boldsymbol{\varphi}_3) = \mathbf{w}'\boldsymbol{\varphi}_3\boldsymbol{\Delta}'\boldsymbol{\tau}$ is the equality

$$(\mathbf{I}_n - \mathbf{P}_{\varphi_3\Delta'})[\boldsymbol{\varphi}_3\mathbf{G}'\mathbf{G}\,\boldsymbol{\varphi}_3(\sigma_A^2 - B_H^{-1}\sigma_B^2) + \boldsymbol{\varphi}_3\,\mathbf{D}'\mathbf{D}\boldsymbol{\varphi}_3(\sigma_B^2 - K_H^{-1}\sigma_U^2) + \boldsymbol{\varphi}_3(\sigma_U^2 + \sigma_e^2)]\mathbf{w} = \mathbf{0}\,.$$

It holds uniformly if and only if the equalities

$$(\mathbf{I}_n - \mathbf{P}_{\varphi_3\Delta'})\boldsymbol{\varphi}_3\mathbf{w} = \mathbf{0}\;, \quad (\mathbf{I}_n - \mathbf{P}_{\varphi_3\Delta'})\boldsymbol{\varphi}_3\mathbf{D}'\mathbf{D}\boldsymbol{\varphi}_3\mathbf{w} = \mathbf{0}$$

and

$$(\mathbf{I}_n - \mathbf{P}_{\varphi_3\Delta'})\,\boldsymbol{\varphi}_3\mathbf{G}'\mathbf{G}\boldsymbol{\varphi}_3\mathbf{w} = \mathbf{0}$$

hold simultaneously. The first equality holds if and only if $\boldsymbol{\varphi}_3\mathbf{w} = \boldsymbol{\varphi}_3\boldsymbol{\Delta}'\mathbf{s}$ for some $\mathbf{s}$, which holds if and only if $\mathbf{D}\boldsymbol{\varphi}_3\mathbf{w} = \mathbf{D}\boldsymbol{\varphi}_3\boldsymbol{\Delta}'\mathbf{s}$ as well as if and only if $\mathbf{G}\boldsymbol{\varphi}_3\mathbf{w} = \mathbf{G}\boldsymbol{\varphi}_3\boldsymbol{\Delta}'\mathbf{s}$ for that $\mathbf{s}$. With this, the remaining two equalities read

$$(\mathbf{I}_n - \mathbf{P}_{\varphi_3\Delta'})\,\boldsymbol{\varphi}_3\mathbf{D}'\mathbf{D}\boldsymbol{\varphi}_3\boldsymbol{\Delta}'\mathbf{s} = \mathbf{0} \quad\text{and}\quad (\mathbf{I}_n - \mathbf{P}_{\varphi_3\Delta'})\,\boldsymbol{\varphi}_3\,\mathbf{G}'\mathbf{G}\boldsymbol{\varphi}_3\boldsymbol{\Delta}'\mathbf{s} = \mathbf{0}\;,$$

which are equivalent to (3.19) and (3.20), respectively, due to the relations $\mathbf{D}'\mathbf{k}^{-\delta}\mathbf{D}\boldsymbol{\varphi}_3 = \boldsymbol{\varphi}_3$, $\mathbf{G}'\mathbf{n}^{-\delta}\mathbf{G}\boldsymbol{\varphi}_3 = \boldsymbol{\varphi}_3$, $\mathbf{D}\boldsymbol{\varphi}_3\mathbf{D}' = \mathbf{K}_0 - \tilde{\mathbf{K}}_0$, $\boldsymbol{\Delta}\boldsymbol{\varphi}_3\mathbf{D}' = \mathbf{N}_0 - \tilde{\mathbf{N}}_0$, $\mathbf{G}\boldsymbol{\varphi}_3\mathbf{G}' = \mathbf{L}_0$, $\boldsymbol{\Delta}\boldsymbol{\varphi}_3\mathbf{G}' = \mathbf{R}_0$ and $\mathbf{C}_3 = \boldsymbol{\Delta}\boldsymbol{\varphi}_3\boldsymbol{\Delta}' = (\mathbf{N}_0 - \tilde{\mathbf{N}}_0)\mathbf{k}^{-\delta}(\mathbf{N}_0 - \tilde{\mathbf{N}}_0)' = \mathbf{R}_0\mathbf{n}^{-\delta}\mathbf{R}_0'$. Finally, the relation between $\mathbf{c}$ and $\mathbf{s}$ follows from the fact that $\mathrm{E}(\mathbf{s}'\boldsymbol{\Delta}\mathbf{y}_3) = \mathbf{s}'\boldsymbol{\Delta}\,\boldsymbol{\varphi}_3\boldsymbol{\Delta}'\boldsymbol{\tau} = \mathbf{c}'\boldsymbol{\tau}$.

Corollary 3.3. For the estimation of $\mathbf{c}'\boldsymbol{\tau} = \mathbf{s}'\mathbf{R}_0\mathbf{n}^{-\delta}\mathbf{R}_0'\boldsymbol{\tau}$ under (3.4) the following applies:

(a) The case $\mathbf{R}_0' = \mathbf{0}$ is to be excluded.

(b) If $\mathbf{R}_0' \neq \mathbf{0}$, then $\mathbf{c}'\boldsymbol{\tau}$ is a contrast, and to satisfy (3.19) and (3.20) by the vector $\mathbf{s}$ it is necessary and sufficient that $(\mathbf{K}_0 - \tilde{\mathbf{K}}_0)(\mathbf{N}_0 - \tilde{\mathbf{N}}_0)'\mathbf{s} \in \mathcal{C}(\mathbf{N}_0' - \tilde{\mathbf{N}}_0')$ and, simultaneously, $\mathbf{L}_0\mathbf{R}_0'\mathbf{s} \in \mathcal{C}(\mathbf{R}_0')$.

(c) If $\mathbf{s}$ is such that $\mathbf{r}'\mathbf{s} = 0$, then the conditions (3.19) and (3.20) can be replaced by

$$(\mathbf{K}_0 - \tilde{\mathbf{K}}_0)(\mathbf{N} - \tilde{\mathbf{N}}_0)'\mathbf{s} = (\mathbf{N}_0 - \tilde{\mathbf{N}}_0)'[(\mathbf{N}_0 - \tilde{\mathbf{N}}_0)\mathbf{k}^{-\delta}(\mathbf{N}_0 - \tilde{\mathbf{N}}_0)']^{-}(\mathbf{N}_0 - \tilde{\mathbf{N}}_0)(\mathbf{N} - \tilde{\mathbf{N}}_0)'\mathbf{s}$$

and

$$\mathbf{L}_0\mathbf{R}'\mathbf{s} = \mathbf{R}_0'(\mathbf{R}_0\mathbf{n}^{-\delta}\mathbf{R}_0')^{-}\mathbf{R}_0\mathbf{R}'\mathbf{s}\;,$$

respectively. To satisfy them it is then necessary and sufficient that both $(\mathbf{K}_0 - \tilde{\mathbf{K}}_0)(\mathbf{N} - \tilde{\mathbf{N}}_0)'\mathbf{s} \in \mathcal{C}(\mathbf{N}_0' - \tilde{\mathbf{N}}_0')$ and $\mathbf{L}_0\mathbf{R}'\mathbf{s} \in \mathcal{C}(\mathbf{R}_0')$.

(d) If all block sizes are equal, i.e., $k_1 = k_2 = \ldots = k_b$, and all superblock sizes are equal, i.e., $n_1 = n_2 = \ldots = n_a$, then the conditions (3.19) and (3.20) are both satisfied automatically by any $\mathbf{s}$.

Proof. The result (a) is obvious, as $\mathbf{R}'_0 = \mathbf{0}$ implies $\mathbf{c} = \mathbf{0}$. To prove (b) note that $\mathbf{R}'_0\mathbf{1}_v = \mathbf{0}$, and that the equations $(\mathbf{N}_0 - \tilde{\mathbf{N}}_0)\mathbf{x}_1 = (\mathbf{K}_0 - \tilde{\mathbf{K}}_0)(\mathbf{N}'_0 - \tilde{\mathbf{N}}'_0)\mathbf{s}$ and $\mathbf{R}'_0\mathbf{x}_2 = \mathbf{L}_0\mathbf{R}'_0\mathbf{s}$ are consistent if and only if (3.19) and (3.20) hold, respectively, since $[(\mathbf{N}_0 - \tilde{\mathbf{N}}_0)\mathbf{k}^{-\delta}(\mathbf{N}_0 - \tilde{\mathbf{N}}_0)']^-(\mathbf{N}_0 - \tilde{\mathbf{N}}_0)\mathbf{k}^{-\delta}$ can be used as a g-inverse of $(\mathbf{N}_0 - \tilde{\mathbf{N}}_0)'$ and $(\mathbf{R}_0\mathbf{n}^{-\delta}\mathbf{R}'_0)^-\mathbf{R}_0\mathbf{n}^{-\delta}$ as a g-inverse of $\mathbf{R}'_0$, and since $(\mathbf{N}_0 - \tilde{\mathbf{N}}_0)\mathbf{k}^{-\delta}(\mathbf{K}_0 - \tilde{\mathbf{K}}_0) = \mathbf{N}_0 - \tilde{\mathbf{N}}_0$ and $\mathbf{R}_0\mathbf{n}^{-\delta}\mathbf{L}_0 = \mathbf{R}_0$. The result (c) is obvious, as $(\mathbf{N}_0 - \tilde{\mathbf{N}}_0)'\mathbf{s} = (\mathbf{N} - \tilde{\mathbf{N}}_0)'\mathbf{s}$ and $\mathbf{R}'_0\mathbf{s} = \mathbf{R}'\mathbf{s}$ if $\mathbf{r}'\mathbf{s} = 0$. The result (d) can easily be checked similarly as Corollary 3.1(d). □

Now it may be noted that if the conditions of Theorem 3.3 are satisfied, then $\widehat{\mathbf{c}'\boldsymbol{\tau}} = \mathbf{s}'\boldsymbol{\Delta}\mathbf{y}_3 = \mathbf{c}'\mathbf{C}_3^-\boldsymbol{\Delta}\boldsymbol{\varphi}_3\mathbf{y}$ is the BLUE of the contrast $\mathbf{c}'\boldsymbol{\tau}$ under (3.4), and that its variance has the form

$$\mathrm{Var}(\widehat{\mathbf{c}'\boldsymbol{\tau}}) = \mathbf{s}'\mathbf{R}_0\mathbf{R}_0\mathbf{s}(\sigma_A^2 - B_H^{-1}\sigma_B^2) + \mathbf{s}'(\mathbf{N}_0 - \tilde{\mathbf{N}}_0)(\mathbf{N}_0 - \tilde{\mathbf{N}}_0)'\mathbf{s}(\sigma_B^2 - K_H^{-1}\sigma_U^2)$$
$$+ \mathbf{s}'\mathbf{R}_0\mathbf{n}^{-\delta}\mathbf{R}'_0\mathbf{s}(\sigma_U^2 + \sigma_e^2) \quad . \tag{3.21}$$

Evidently, if $k_1 = k_2 = \ldots = k_b = k$ and $n_1 = n_2 = \ldots = n_a = n_0$ $(=n/a)$, then the variance (3.21) reduces to

$$\mathrm{Var}(\widehat{\mathbf{c}'\boldsymbol{\tau}}) = n_0^{-1}\mathbf{s}'\mathbf{R}_0\mathbf{R}'_0\mathbf{s}[n_0\sigma_A^2 + (k - B_H^{-1}n_0)\,\sigma_B^2 + (1 - K_H^{-1}k)\,\sigma_U^2 + \sigma_e^2]$$
$$= \mathbf{c}'(n_0^{-1}\mathbf{R}_0\mathbf{R}'_0)^-\,\mathbf{c}[n_0\sigma_A^2 + (k - B_H^{-1}n_0)\,\sigma_B^2 + (1 - K_H^{-1}k)\,\sigma_U^2 + \sigma_e^2]. \tag{3.22}$$

It reduces further to $\mathrm{Var}(\widehat{\mathbf{c}'\boldsymbol{\tau}}) = \mathbf{c}'(n_0^{-1}\mathbf{R}_0\mathbf{R}'_0)^-\mathbf{c}(n_0\sigma_A^2 + \sigma_e^2)$ if $k = B_H^{-1}n_0 = K_H$, i.e., if the number of available blocks in each superblock is constant, equal to n_0/k $(= b/a)$, and the size of each available block is constant, equal to k.

An answer to the question what is necessary and sufficient for the conditions of Theorem 3.3 to be satisfied by any $\mathbf{s}$, can be given as follows.

Corollary 3.4. The conditions (3.19) and (3.20) hold for any $\mathbf{s}$, i.e., the equalities

$$(\mathbf{K}_0 - \tilde{\mathbf{K}}_0)(\mathbf{N}_0 - \tilde{\mathbf{N}}_0)' = (\mathbf{N}_0 - \tilde{\mathbf{N}}_0)'[(\mathbf{N}_0 - \tilde{\mathbf{N}}_0)\mathbf{k}^{-\delta}(\mathbf{N}_0 - \tilde{\mathbf{N}}_0)']^-(\mathbf{N}_0 - \tilde{\mathbf{N}}_0)(\mathbf{N}_0 - \tilde{\mathbf{N}}_0)' \tag{3.23}$$

and

$$\mathbf{L}_0\mathbf{R}'_0 = \mathbf{R}'_0(\mathbf{R}_0\mathbf{n}^{-\delta}\mathbf{R}'_0)^-\,\mathbf{R}_0\mathbf{R}'_0 \tag{3.24}$$

hold, if and only if for any $b\times 1$ vector $\mathbf{t}$ that satisfies the equality $(\mathbf{N}_0 - \tilde{\mathbf{N}}_0)\mathbf{t} = \mathbf{0}$ the equality $(\mathbf{N}_0 - \tilde{\mathbf{N}}_0)(\mathbf{K}_0 - \tilde{\mathbf{K}}_0)\mathbf{t} = \mathbf{0}$ holds too, and for any $a\times 1$ vector $\mathbf{u}$ that satisfies the equality $\mathbf{R}_0\mathbf{u} = \mathbf{0}$ the equality $\mathbf{R}_0\mathbf{L}_0\mathbf{u} = \mathbf{0}$ also holds.

Proof. The proof follows the same pattern as that of Corollary 3.2 in Caliński and Kageyama (1991). □

Remark 3.4. (a) Due to the relations

$$\mathbf{N}_0 - \tilde{\mathbf{N}}_0 = \mathbf{R}_0\mathbf{n}^{-\delta}\mathbf{G}\mathbf{D}', \quad \mathbf{K}_0 - \tilde{\mathbf{K}}_0 = \mathbf{D}\mathbf{G}'(\mathbf{I}_a - n^{-1}\mathbf{1}_a\mathbf{n}')\mathbf{n}^{-\delta}\mathbf{G}\mathbf{D}'$$

and

$$\mathbf{R}_0 = \mathbf{R}(\mathbf{I}_a - n^{-1}\mathbf{1}_a\mathbf{n}'),$$

the conditions given in Corollary 3.4 for the equalities (3.23) and (3.24) can be reduced as follows. The equalities (3.23) and (3.24) hold simultaneously if and only if for any (nonzero) vector that is $\mathbf{n}^{\delta}$-orthogonal to $\mathbf{1}_a$, $\mathbf{u}_0$ say (i.e. such that $\mathbf{n}'\mathbf{u}_0 = 0$), the equality $\mathbf{R}\mathbf{u}_0 = \mathbf{0}$ implies the equalities

$$(\mathbf{N}_0 - \tilde{\mathbf{N}}_0)\mathbf{D}\mathbf{G}'\mathbf{u}_0 = \mathbf{0} \quad \text{and} \quad \mathbf{R}_0\mathbf{n}^{\delta}\mathbf{u}_0 = \mathbf{0}\,.$$

(b) If $\text{rank}(\mathbf{R}_0) = a - 1$, i.e. the columns of the matrix $\mathbf{R}$ are linearly independent [as $\text{rank}(\mathbf{R}_0) = \text{rank}(\mathbf{R}) - 1$], then a nonzero vector $\mathbf{u}$ that satisfies $\mathbf{R}_0\mathbf{u} = \mathbf{0}$ must be equal or proportional to $\mathbf{1}_a$ [i.e., $\mathbf{u} \in C(\mathbf{1}_a)$], and so satisfy also the equality $\mathbf{R}_0\mathbf{L}_0\mathbf{u} = \mathbf{0}$. Moreover, with this rank of $\mathbf{R}_0$, any vector $\mathbf{t}$ such that $\mathbf{n}^{-\delta}\mathbf{G}\mathbf{D}'\mathbf{t} \in C(\mathbf{1}_a)$ satisfies simultaneously the equalities $(\mathbf{N}_0 - \tilde{\mathbf{N}}_0)\mathbf{t} = \mathbf{0}$ and $(\mathbf{N}_0 - \tilde{\mathbf{N}}_0)(\mathbf{K}_0 - \tilde{\mathbf{K}}_0)\mathbf{t} = \mathbf{0}$. Any other $\mathbf{t}$ for which the first of these two holds has to be such that $\mathbf{G}\mathbf{D}'\mathbf{t} = \mathbf{0}$ and then the second equality also holds. Thus, the conditions of Corollary 3.4 are then satisfied automatically, whatever the k_j's and n_h's are.

Remark 3.5. From the proof of Theorem 3.3 it is evident that the equalities (3.19) and (3.20) hold simultaneously for any $\mathbf{s}$, i.e. (3.23) and (3.24) hold, if and only if $\text{Cov}(\mathbf{y}_3)\boldsymbol{\varphi}_3\boldsymbol{\Delta}' = \mathbf{P}_{\boldsymbol{\varphi}_3\boldsymbol{\Delta}'}\text{Cov}(\mathbf{y}_3)\boldsymbol{\varphi}_3\boldsymbol{\Delta}'$, which shows that not only $\boldsymbol{\varphi}_3\boldsymbol{\Delta}'\mathbf{s}$ but also $\text{Cov}(\mathbf{y}_3)\boldsymbol{\varphi}_3\boldsymbol{\Delta}'\mathbf{s}$ belongs to $C(\boldsymbol{\varphi}_2\boldsymbol{\Delta}')$ for any $\mathbf{s}$, and thus the condition stated in Theorem 4 of Zyskind (1967), when applied to Theorem 3.3, is satisfied. This means that the BLUE of any function $\mathbf{c}'\boldsymbol{\tau}$, when $\mathbf{c} \in C(\mathbf{C}_3)$, obtainable under the inter-superblock submodel (3.4) is simultaneously the SLSE.

Remark 3.5 implies in particular that if the equalities (3.23) and (3.24) hold, then the inter-superblock analysis of variance is obtainable in the form

$$\mathbf{y}'\boldsymbol{\varphi}_3\mathbf{y} = \mathbf{Q}_3'\mathbf{C}_3^-\mathbf{Q}_3 + \mathbf{y}'\boldsymbol{\psi}_3\mathbf{y}\ ,$$

where $\mathbf{Q}_3'\mathbf{C}_3^-\mathbf{Q}_3$, with $\mathbf{Q}_3 = \boldsymbol{\Delta}\boldsymbol{\varphi}_3\mathbf{y}$, is the inter-superblock treatment sum of squares, and $\mathbf{y}'\boldsymbol{\psi}_3\mathbf{y}$, with $\boldsymbol{\psi}_3 = \boldsymbol{\varphi}_3 - \boldsymbol{\varphi}_3\boldsymbol{\Delta}'\mathbf{C}^-\boldsymbol{\Delta}\boldsymbol{\varphi}_3 = \boldsymbol{\varphi}_3(\mathbf{I}_n - \boldsymbol{\Delta}'\mathbf{C}_3^-\boldsymbol{\Delta})\,\boldsymbol{\varphi}_3$, is the inter-superblock residual sum of squares, the first on $h_3 = \text{rank}(\mathbf{C}_3) = \text{rank}(\mathbf{R}_0)$ d.f., the second on

$a-1-h_3 = \text{rank}(\boldsymbol{\psi}_3)$ d.f. The resulting inter-superblock residual mean square $s_3 = \mathbf{y}'\boldsymbol{\psi}_3\mathbf{y} \,/\,(a-1-h_3)$ is the MINQUE of

$$\begin{aligned}\sigma_3^2 &= (a-1-h_3)^{-1}\{\text{tr}(\mathbf{L}_0) - \text{tr}[\mathbf{R}_0'(\mathbf{R}_0\mathbf{n}^{-\delta}\mathbf{R}_0')^{-}\mathbf{R}_0]\}(\sigma_A^2 - B_H^{-1}\sigma_B^2)\\ &+ (a-1-h_3)^{-1}\{\text{tr}(\mathbf{K}_0 - \tilde{\mathbf{K}}_0)\\ &- \text{tr}[(\mathbf{N}_0-\tilde{\mathbf{N}}_0)'[(\mathbf{N}_0-\tilde{\mathbf{N}}_0)\mathbf{k}^{-\delta}(\mathbf{N}_0-\tilde{\mathbf{N}}_0)']^{-}(\mathbf{N}_0-\tilde{\mathbf{N}}_0)\}(\sigma_B^2 - K_H^{-1}\sigma_U^2) + \sigma_U^2 + \sigma_e^2.\end{aligned} \tag{3.25}$$

In case of

$$k_1 = k_2 = \ldots = k_b = k \quad \text{and} \quad n_1 = n_2 = \ldots = n_a = n_0\,, \tag{3.26}$$

the estimated variance parametric function (3.25) becomes

$$\sigma_3^2 = n_0\sigma_A^2 + (k - B_H^{-1}n_0)\,\sigma_B^2 + (1 - K_H^{-1}k)\,\sigma_U^2 + \sigma_e^2\,,$$

further reducing to $n_0\sigma_A^2 + \sigma_e^2$ if $B_H^{-1}n_0 = k = K_H$, i.e. $B_H = n_0/k\ (= b/a)$. It should be noticed, however, that $a-1-h_3 = 0$ if $a-1 = h_3$ (that $a-1 \geq h_3$ is obvious). In that case no estimator for σ_3^2 exists in the inter-superblock analysis.

Thus, in the case of equal k_j's, equal n_h's and $a-1 > h_3$, the mean square s_3^2 can be used to obtain an unbiased estimator of the variance (3.22), in the form

$$\widehat{\text{Var}}(\mathbf{c}'\hat{\boldsymbol{\tau}}) = n_0^{-1}\mathbf{s}'\mathbf{R}_0\mathbf{R}_0'\,\mathbf{s}\,s_3^2 = n_0\mathbf{c}'(\mathbf{R}_0\mathbf{R}_0')^{-}\mathbf{c}\,s_3^2\,.$$

In general, the estimation of (3.21) is not so simple.

Furthermore, if the equalities (3.26) hold, then $\text{Cov}(\mathbf{y}_3) = \boldsymbol{\varphi}_3\sigma_3^2$ and, similarly as for the intra-block analysis, it can be shown that under the multivariate normality assumption the quadratic functions $\mathbf{Q}_3'\mathbf{C}_3^{-}\mathbf{Q}_3/\sigma_3^2$ and $\mathbf{y}'\boldsymbol{\psi}_3\mathbf{y}/\sigma_3^2$ have independent χ^2 distributions, the first being non-central with h_3 d.f. and with the non-centrality parameter $\delta_3 = \boldsymbol{\tau}'\mathbf{C}_3\boldsymbol{\tau}/\sigma_3^2$, the second central with $a-1-h_3$ d.f. Hence, the hypothesis $\boldsymbol{\tau}'\mathbf{C}_3\boldsymbol{\tau} = 0$, equivalent to $\text{E}(\mathbf{y}_3) = 0$ [or $\boldsymbol{\varphi}_3\boldsymbol{\Delta}'\boldsymbol{\tau} = \mathbf{0}$, or $\mathbf{P}_{G'}\text{E}(\mathbf{y}) \in C(\mathbf{1}_n)$], can be tested by the variance ratio criterion

$$h_3^{-1}\mathbf{Q}_3'\mathbf{C}_3^{-}\mathbf{Q}_3\,/\,s_3^2\,,$$

which under the assumed normality has then the F distribution with h_3 and $a-1-h_3$ d.f., central when the hypothesis is true. This, however, does not apply to the general case, when the equalities (3.26) do not hold.

3.4. *The total area submodel.*

Considering the fourth submodel, (3.5), it is evident that its properties are

$$\text{E}(\mathbf{y}_4) = \boldsymbol{\varphi}_4\boldsymbol{\Delta}'\boldsymbol{\tau} = n^{-1}\mathbf{1}_n\mathbf{r}'\boldsymbol{\tau}$$

and

$$\text{Cov}(\mathbf{y}_4) = \varphi_4[(n^{-1}\mathbf{n}'\mathbf{n} - N_A^{-1}n)\,\sigma_A^2 + (n^{-1}\mathbf{k}'\mathbf{k} - B_H^{-1}n^{-1}\mathbf{n}'\mathbf{n})\,\sigma_B^2$$
$$+ (1 - K_H^{-1}n^{-1}\mathbf{k}'\mathbf{k})\,\sigma_U^2 + \sigma_e^2]\ ,$$

the latter reducing in the case of equal k_j's and equal n_h's [i.e. in the case of (3.26)] to

$$\text{Cov}(\mathbf{y}_4) = \varphi_4[(n_0 - N_A^{-1}n)\,\sigma_A^2 + (k - B_H^{-1}n_0)\,\sigma_B^2 + (1 - K_H^{-1}k)\,\sigma_U^2 + \sigma_e^2]\ .$$

In the general case the following main result concerning estimation under (3.5) is obtainable.

Theorem 3.4. Under (3.5), a function $\mathbf{w}'\mathbf{y}_4 = \mathbf{w}'\varphi_4\mathbf{y}$ is uniformly the BLUE of $\mathbf{c}'\boldsymbol{\tau}$ if and only if $\varphi_4\mathbf{w} = \varphi_4\boldsymbol{\Delta}'\mathbf{s}$, where the vectors $\mathbf{c}$ and $\mathbf{s}$ are in the relation $\mathbf{c} = \boldsymbol{\Delta}\varphi_4\boldsymbol{\Delta}'\mathbf{s}$ $(= n^{-1}\mathbf{r}\mathbf{r}'\mathbf{s})$.

Proof. This can be proved following exactly the same pattern as that in the proof of Theorem 3.1 of Caliński and Kageyama (1991, Section 3.1). □

Remark 3.6. (a) The only parametric functions for which the BLUEs under (3.5) exist are those defined as $\mathbf{c}'\boldsymbol{\tau} = (\mathbf{s}'\mathbf{r})n^{-1}\mathbf{r}'\boldsymbol{\tau}$, i.e. the general parametric mean and its multiplicities, contrast being excluded a fortiori (as $\mathbf{1}_v'\mathbf{c} = \mathbf{r}'\mathbf{s}$).

(b) Since

$$\text{Cov}(\mathbf{y}_4)\varphi_4\boldsymbol{\Delta}' = \varphi_4\boldsymbol{\Delta}'[(n^{-1}\mathbf{n}'\mathbf{n} - N_A^{-1}n)\,\sigma_A^2 + (n^{-1}\mathbf{k}'\mathbf{k} - B_H^{-1}n^{-1}\mathbf{n}'\mathbf{n})\,\sigma_B^2$$
$$+ (1 - K_H^{-1}n^{-1}\mathbf{k}'\mathbf{k})\,\sigma_e^2]\ ,$$

the BLUEs under (3.5) and the SLSEs are the same (on account of Zyskind's, 1967, Theorem 4, applied to Theorem 3.4).

If $\mathbf{c}'\boldsymbol{\tau} = (\mathbf{s}'\mathbf{r})n^{-1}\mathbf{r}'\boldsymbol{\tau} = (\mathbf{c}'\mathbf{1}_v)n^{-1}\mathbf{r}'\boldsymbol{\tau}$, then the variance of its BLUE under (3.5), i.e. of $\widehat{\mathbf{c}'\boldsymbol{\tau}} = \mathbf{s}'\boldsymbol{\Delta}\mathbf{y}_4 = \mathbf{c}'(\boldsymbol{\Delta}\,\varphi_4\boldsymbol{\Delta}')^{-}\boldsymbol{\Delta}\,\varphi_4\mathbf{y}$, is of the form

$$\text{Var}(\widehat{\mathbf{c}'\boldsymbol{\tau}}) = \mathbf{s}'\boldsymbol{\Delta}\varphi_4\boldsymbol{\Delta}'\mathbf{s}[(n^{-1}\mathbf{n}'\mathbf{n} - N_A^{-1}n)\sigma_A^2 + (n^{-1}\mathbf{k}'\mathbf{k} - B_H^{-1}n^{-1}\mathbf{n}'\mathbf{n})\,\sigma_B^2$$
$$+ (1 - K_H^{-1}n^{-1}\mathbf{k}'\mathbf{k})\,\sigma_U^2 + \sigma_e^2]$$
$$= n^{-1}(\mathbf{c}'\mathbf{1}_v)^2[(n^{-1}\mathbf{n}'\mathbf{n} - N_A^{-1}n)\,\sigma_A^2 + (n^{-1}\mathbf{k}'\mathbf{k} - B_H^{-1}n^{-1}\mathbf{n}'\mathbf{n})\,\sigma_B^2$$
$$+ (1 - K_H^{-1}n^{-1}\mathbf{k}'\mathbf{k})\,\sigma_U^2 + \sigma_e^2]\ . \tag{3.27}$$

Evidently, if the equalities (3.26) hold, then the variance (3.27) reduces to

$$\text{Var}(\widehat{\mathbf{c}'\boldsymbol{\tau}}) = n^{-1}(\mathbf{c}'\mathbf{1}_v)^2[(1 - N_A^{-1}a)n_0\sigma_A^2 + (1 - B_H^{-1}k^{-1}n_0)k\,\sigma_B^2$$
$$+ (1 - K_H^{-1}k)\,\sigma_U^2 + \sigma_e^2]\ , \tag{3.28}$$

and if, in addition, $a = N_A$, $B_H = n_0/k$ $(= b/a)$, $K_H = k$, which may be considered as the usual case, then

$$\mathrm{Var}(\mathbf{c}'\hat{\boldsymbol{\tau}}) = n^{-1}(\mathbf{c}'\mathbf{1}_v)^2\sigma_e^2 \,. \tag{3.29}$$

Finally, it may be noted that since $\mathbf{P}_{\varphi_4\Delta'} = \varphi_4$ (as $n^{-1}\mathbf{1}_v\mathbf{1}_v'$ is a g-inverse of $\Delta\varphi_4\Delta' = n^{-1}\mathbf{r}\mathbf{r}'$) and, hence,

$$(\mathbf{I}_n - \mathbf{P}_{\varphi_4\Delta'})\mathbf{y}_4 = \mathbf{0} \;\text{ and }\; (\mathbf{I}_n - \mathbf{P}_{\varphi_4\Delta'})\mathrm{Cov}(\mathbf{y}_4) = \mathbf{0}\,,$$

the vector $\mathbf{P}_{\varphi_4\Delta'}\mathbf{y}_4 = \mathbf{y}_4 = n^{-1}\mathbf{1}_n\mathbf{1}_n'\mathbf{y}$ is itself the BLUE of its expectation, $n^{-1}\mathbf{1}_n\mathbf{r}'\boldsymbol{\tau}$, leaving no residuals.

3.5. Some special cases

It follows from the considerations above that any function $\mathbf{s}'\Delta\mathbf{y}$ can be resolved into four components in the form

$$\mathbf{s}'\Delta\mathbf{y} = \mathbf{s}'\mathbf{Q}_1 + \mathbf{s}'\mathbf{Q}_2 + \mathbf{s}'\mathbf{Q}_3 + \mathbf{s}'\mathbf{Q}_4\,, \tag{3.30}$$

with $\mathbf{Q}_1 = \Delta\mathbf{y}_1 = \Delta\varphi_1\mathbf{y}$, $\mathbf{Q}_2 = \Delta\mathbf{y}_2 = \Delta\varphi_2\mathbf{y}$, $\mathbf{Q}_3 = \Delta\mathbf{y}_3 = \Delta\varphi_3\mathbf{y}$, $\mathbf{Q}_4 = \Delta\mathbf{y}_4 = \Delta\varphi_4\mathbf{y}$. Each of the components in (3.30) represents a contribution to $\mathbf{s}'\Delta\mathbf{y}$ from a different stratum. Similarly as in Caliński and Kageyama (1991, Section 3.4), the components $\mathbf{s}'\mathbf{Q}_1$, $\mathbf{s}'\mathbf{Q}_2$, $\mathbf{s}'\mathbf{Q}_3$, and $\mathbf{s}'\mathbf{Q}_4$ may then be called the intra-block, the inter-block-intra-superblock, the inter-superblock and the total-area component, respectively.

In connection with formula (3.30) it is interesting to consider some special cases of the vector $\mathbf{s}$ (and hence of $\mathbf{c} = \mathbf{r}^\delta\mathbf{s}$), similarly as in Section 3.4 of Caliński and Kageyama (1991). As the first case suppose that $\mathbf{N}'\mathbf{s} = \mathbf{0}$, i.e., $\mathbf{s}$ is orthogonal to the columns of $\mathbf{N}$ [or, equivalently, that $\Delta'\mathbf{s} \in \mathcal{C}^\perp(\mathbf{D}')$], which also implies that $\mathbf{r}'\mathbf{s} = 0$ (i.e. $\mathbf{1}_v'\mathbf{c} = 0$). Then $\mathbf{s}'\mathbf{Q}_2 = 0$ and $\mathbf{s}'\mathbf{Q}_3 = -\mathbf{s}'\mathbf{Q}_4$, giving the equality $\mathbf{s}'\Delta\mathbf{y} = \mathbf{s}'\mathbf{Q}_1$. Thus, in this case, only the intra-block stratum contributes. As the second case, suppose that $\mathbf{s}$ is such that $\mathbf{N}'\mathbf{s} \neq \mathbf{0}$ but the conditions $\varphi_1\Delta'\mathbf{s} = \mathbf{0}$ and $\mathbf{G}\Delta'\mathbf{s} = \mathbf{0}$ are satisfied [or equivalently, that $\Delta'\mathbf{s} \in \mathcal{C}^\perp(\mathbf{G}') \cap \mathcal{C}(\mathbf{D}')$], which also implies that $\mathbf{R}'\mathbf{s} = \mathbf{0}$. Then $\mathbf{s}'\mathbf{Q}_1 = 0$ and $\mathbf{s}'\mathbf{Q}_3 = -\mathbf{s}'\mathbf{Q}_4$, which implies that $\mathbf{s}'\Delta\mathbf{y} = \mathbf{s}'\Delta\varphi_2\mathbf{y}$, showing that the contribution comes from the inter-block-intra-superblock stratum only. As the third case suppose that $\mathbf{s}$ is such that $\mathbf{R}'\mathbf{s} \neq \mathbf{0}$ but it satisfies the conditions $\varphi_1\Delta'\mathbf{s} = \mathbf{0}$, $\varphi_2\Delta'\mathbf{s} = \mathbf{0}$ and $\mathbf{1}_n'\Delta'\mathbf{s} = 0$ [or, equivalently, that $\Delta'\mathbf{s} \in \mathcal{C}^\perp(\mathbf{1}_n) \cap \mathcal{C}(\mathbf{G}')$], which also implies that $\mathbf{r}'\mathbf{s} = 0$ (i.e., $\mathbf{1}_v'\mathbf{c} = 0$). Then $\mathbf{s}'\mathbf{Q}_1 = \mathbf{s}'\mathbf{Q}_2 = \mathbf{s}'\mathbf{Q}_4 = 0$, giving the equality $\mathbf{s}'\Delta\mathbf{y} = \mathbf{s}'\mathbf{Q}_3$. Thus, in this case, the contribution comes from the inter-superblock stratum only. Finally, suppose that $\mathbf{s} \in \mathcal{C}(\mathbf{1}_v)$, i.e., that $\mathbf{s}$ is proportional to the vector $\mathbf{1}_v$ [or, equivalently, that $\Delta'\mathbf{s} \in \mathcal{C}(\mathbf{1}_n)$]. Then, on account of (3.7), $\mathbf{s}'\mathbf{Q}_1 = \mathbf{s}'\mathbf{Q}_2 = \mathbf{s}'\mathbf{Q}_3 = 0$, giving

$\mathbf{s}'\Delta\mathbf{y} = \mathbf{s}'\mathbf{Q}_4$. This means that the only contribution is then from the total-area stratum.

Applying to the above cases the results of Section 2.2, one can prove the following.

Corollary 3.5. The function $\mathbf{s}'\Delta\mathbf{s}$ is the BLUE of $\mathbf{c}'\boldsymbol{\tau} = \mathbf{s}'\mathbf{r}^{\delta}\boldsymbol{\tau}$ under the overall model (2.7) in the following four cases:

(a) $\mathbf{N}'\mathbf{s} = \mathbf{0}$ (implying $\mathbf{r}'\mathbf{s} = 0$); the BLUE is then equal to $\mathbf{s}'\mathbf{Q}_1$ and its variance is of the form

$$\mathrm{Var}(\mathbf{c}'\hat{\boldsymbol{\tau}}) = \mathbf{s}'\mathbf{r}^{\delta}\mathbf{s}(\sigma_U^2 + \sigma_e^2) = \mathbf{c}'\mathbf{r}^{-\delta}\mathbf{c}(\sigma_U^2 + \sigma_e^2)\,. \tag{3.31}$$

(b) $\mathbf{N}'\mathbf{s} \neq \mathbf{0}$, but $\boldsymbol{\varphi}_1\Delta'\mathbf{s} = \mathbf{0}$ and $\mathbf{R}'\mathbf{s} = \mathbf{0}$, provided that the first part of the condition (ii) of Theorem 2.2 holds with regard to those connected subdesigns of D^* to which the nonzero elements of $\mathbf{s}$ correspond; the BLUE is then equal to $\mathbf{s}'\mathbf{Q}_2$ and its variance is of the form

$$\mathrm{Var}(\mathbf{c}'\hat{\boldsymbol{\tau}}) = \mathbf{s}'\mathbf{N}\mathbf{N}'\mathbf{s}(\sigma_B^2 - K_H^{-1}\sigma_U^2) + \mathbf{s}'\mathbf{r}^{\delta}\mathbf{s}(\sigma_U^2 + \sigma_e^2)\,. \tag{3.32}$$

(c) $\mathbf{R}'\mathbf{s} \neq \mathbf{0}$, but $\boldsymbol{\varphi}_1\Delta'\mathbf{s} = \mathbf{0}$, $\boldsymbol{\varphi}_2\Delta'\mathbf{s} = \mathbf{0}$ and $\mathbf{r}'\mathbf{s} = 0$, provided that the whole condition (ii) of Theorem 2.2 holds with regard to those connected subdesigns of D^* and D to which the nonzero elements of $\mathbf{s}$ correspond; the BLUE is then equal to $\mathbf{s}'\mathbf{Q}_3$ and its variance is of the form

$$\mathrm{Var}(\mathbf{c}'\hat{\boldsymbol{\tau}}) = \mathbf{s}'\mathbf{R}\mathbf{R}'\mathbf{s}(\sigma_A^2 - B_H^{-1}\sigma_B^2) + \mathbf{s}'(\mathbf{N} - \tilde{\mathbf{N}}_0)(\mathbf{N} - \tilde{\mathbf{N}}_0)'\mathbf{s}(\sigma_B^2 - K_H^{-1}\sigma_U^2)$$
$$+ \mathbf{s}'\mathbf{r}^{\delta}\mathbf{s}(\sigma_U^2 + \sigma_e^2)\,. \tag{3.33}$$

(d) $\mathbf{s} = n^{-1}(\mathbf{s}'\mathbf{r})\mathbf{1}_v = n^{-1}(\mathbf{c}'\mathbf{1}_v)\mathbf{1}_v$, provided that the whole condition (ii) of Theorem 2.2 holds; the BLUE is then equal to $\mathbf{s}'\mathbf{Q}_4$ and its variance is of the form (3.27).

Proof. These results can be proved in a similar way as those in Corollary 3.3 of Caliński and Kageyama (1991). See also Remark 3.7 and 3.8 there. □

Remark 3.7. For the case (b) of Corollary 3.5 it should be noted that the conditions $\boldsymbol{\varphi}_1\Delta'\mathbf{s} = \mathbf{0}$ and $\mathbf{R}'\mathbf{s} = \mathbf{0}$ (i.e. $\mathbf{r}_h'\mathbf{s} = 0$ for all h) imply that the design D^* is disconnected, and also any component design D_h (h=1,2,…,a) is disconnected or such in which treatments to which nonzero elements of $\mathbf{s}$ correspond are not present (i.e., the corresponding rows of $\mathbf{N}_h$ are void).

Remark 3.8. For the case (c) of Corollary 3.5 it should be noted that the conditions $\boldsymbol{\varphi}_1\Delta'\mathbf{s} = \mathbf{0}$, $\boldsymbol{\varphi}_2\Delta'\mathbf{s} = \mathbf{0}$ and $\mathbf{r}'\mathbf{s} = 0$ imply that the design D^* is disconnected, and also the design D is disconnected.

Also note that if $k_1 = k_2 = \ldots = k_b = k$, then the formula (3.32) reduces to

$$\mathrm{Var}(\mathbf{c}'\hat{\boldsymbol{\tau}}) = \mathbf{c}'\mathbf{r}^{-\delta}\mathbf{c}[k\sigma_B^2 + (1 - K_H^{-1}k)\,\sigma_U^2 + \sigma_e^2]\,.$$

If, in addition to the equality of all k_j's, also $n_1 = n_2 = \ldots = n_a = n_0$, then the formula (3.33) reduces to

$$\mathrm{Var}(\mathbf{c}'\hat{\boldsymbol{\tau}}) = \mathbf{c}'\mathbf{r}^{-\delta}\mathbf{c}[n_0\sigma_A^2 + (k - B_H^{-1}n_0)\,\sigma_B^2 + (1 - K_H^{-1}k)\,\sigma_U^2 + \sigma_e^2].$$

Finally, if all k_j's are equal and all n_h's are equal, then the variance given in Corollary 3.5 for the case (d) reduces from (3.27) to (3.28), or possibly to (3.29).

4. Examples

The theory presented in the previous sections will now be illustrated by some examples.

Example 4.1. Let the block design considered as Example 3.1 in Caliński and Kageyama (1991) [taken from Pearce (1983, p. 102)] be redefined as $\mathbf{N} = [\mathbf{N}_1 : \mathbf{N}_2 : \mathbf{N}_3]$, where

Superblock 1 *Superblock 2* *Superblock 3*

$$\mathbf{N}_1 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \qquad \mathbf{N}_2 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{N}_3 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix},$$

i.e. as a nested block design with three superblocks ($a = 3$), each composed of two blocks ($b_1 = b_2 = b_3 = 2$). The blocks within a superblock are of equal sizes, $k_{1(1)} = k_{2(1)} = 4$, $k_{1(2)} = k_{2(2)} = 5$ and $k_{1(3)} = k_{2(3)} = 6$. While the incidence submatrices $\mathbf{N}_1$, $\mathbf{N}_2$, $\mathbf{N}_3$ describe the component designs within the three superblocks, D_1, D_2, D_3, the design for superblocks, D, is described by the incidence matrix

$$\mathbf{R} = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}.$$

Evidently, the columns of $\mathbf{R}$ are the vectors of treatment replications in the component designs. The treatment replications for the whole design are then given by the vector $\mathbf{r} = [6, 6, 6, 6, 3, 3]'$ $(= \mathbf{R}\mathbf{1}_3)$, while the superblock sizes are given by the vector $\mathbf{n} = [8, 10, 12]'$ $(= \mathbf{R}'\mathbf{1}_6)$. Suppose that this design is applied to available experimental units grouped into blocks and those into superblocks, all of conformable sizes, to allow for the threefold randomization performed as described in Section 2.

To see for which contrasts the BLUEs under the intra-block submodel exist, it may be helpful to find the matrix $\mathbf{C}_1 = \mathbf{r}^{\delta} - \mathbf{N}\mathbf{k}^{-\delta}\mathbf{N}'$. Here

$$\mathbf{C}_1 = \frac{1}{30}\begin{bmatrix} 143 & -37 & -37 & -37 & -16 & -16 \\ -37 & 143 & -37 & -37 & -16 & -16 \\ -37 & -37 & 143 & -37 & -16 & -16 \\ -37 & -37 & -37 & 143 & -16 & -16 \\ -16 & -16 & -16 & -16 & 74 & -10 \\ -16 & -16 & -16 & -16 & -10 & 74 \end{bmatrix}.$$

Evidently, $\mathrm{rank}(\mathbf{C}_1) = v - 1 = 5$, which also follows directly from the fact that the design is connected (see , e.g., Caliński, 1993, Lemma 2.2). This implies that the columns of $\mathbf{C}_1$ span the subspace of all contrasts (of all vectors $\mathbf{c}$ representing contrasts). Hence, on account of Theorem 3.1 of Caliński and Kageyama (1991), for any contrast $\mathbf{c}'\boldsymbol{\tau}$ there exists the BLUE under the intra-block submodel. It is of the form $\widehat{\mathbf{c}'\boldsymbol{\tau}} = \mathbf{s}'\boldsymbol{\Delta}\mathbf{y}_1$, where $\mathbf{s}$ is such that $\mathbf{c} = \mathbf{C}_1\mathbf{s}$. In particular, to estimate the contrast between treatment 1 and treatment 2 one can use the vector $\mathbf{s} = (1/6)[1, -1, 0, 0, 0, 0]'$. It should, however, be noted that for this $\mathbf{s}$ the equality $\mathbf{N}'\mathbf{s} = \mathbf{0}$ holds, which on account of Corollary 3.5(a) implies that $\mathbf{s}'\boldsymbol{\Delta}\mathbf{y}_1 = \mathbf{s}'\boldsymbol{\Delta}\mathbf{y}$, and this function is the BLUE of $\mathbf{c}'\dot{\boldsymbol{\tau}}$ under the overall model (2.7). The same is true for any contrast among the first four treatments, as can easily be seen from the form of the matrix $\mathbf{N}$ of the considered design.

As to the contrasts for which the BLUEs under the inter-block-intra-superblock submodel may possibly exist, they are to be searched in view of the matrix $\mathbf{C}_2 = \tilde{\mathbf{N}}_0\mathbf{k}^{-\delta}\tilde{\mathbf{N}}_0'$ (see Theorem 3.2). For that note that in the present example

$$\tilde{\mathbf{N}}_0 = [\mathbf{N}_{01} : \mathbf{N}_{02} : \mathbf{N}_{03}] = [\mathbf{0} : \mathbf{N}_{02} : \mathbf{0}] ,$$

with

$$\mathbf{N}_{02} = \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix}'$$

from which

$$\mathbf{C}_2 = \frac{1}{10}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix}, \text{ of rank 1.}$$

This shows that there is only one contrast for which the BLUE under the considered submodel may exist, the contrast between treatment 5 and treatment 6. It can be represented by the vector $\mathbf{c} = \mathbf{C}_2\mathbf{s}$, where $\mathbf{s} = 10[0, 0, 0, 0, 1, 0]'$. Since the block sizes within the second superblock are equal ($k_{1(2)} = k_{2(2)} = 5$), this being the only superblock which contributes to $\tilde{\mathbf{N}}_0$ and hence to $\mathbf{C}_2$, the BLUE under the inter-block-intra-superblock submodel really exists for this contrast, as it follows from Corollary 3.2 and Remark 3.1. It has the form $\widehat{\mathbf{c}'\boldsymbol{\tau}} = \mathbf{s}'\boldsymbol{\Delta}\mathbf{y}_2$.

Turning now to the estimation of contrasts under the inter-superblock submodel, it should be recalled (from Theorem 3.3) that the BLUEs may exist only for such contrasts which are generated by the matrix $\mathbf{C}_3 = \boldsymbol{\Delta}\,\boldsymbol{\varphi}_3\boldsymbol{\Delta}' = (\mathbf{N}_0 - \tilde{\mathbf{N}}_0)\mathbf{k}^{-\delta}(\mathbf{N}_0 - \tilde{\mathbf{N}}_0)'$. Since here

$$\mathbf{N}_0 - \tilde{\mathbf{N}}_0 = \frac{1}{5}\begin{bmatrix} 1 & 1 & 0 & 0 & -1 & -1 \\ 1 & 1 & 0 & 0 & -1 & -1 \\ 1 & 1 & 0 & 0 & -1 & -1 \\ 1 & 1 & 0 & 0 & -1 & -1 \\ -2 & -2 & 0 & 0 & 2 & 2 \\ -2 & -2 & 0 & 0 & 2 & 2 \end{bmatrix},$$

it follows that

$$\mathbf{C}_3 = \frac{1}{30}\begin{bmatrix} 1 & 1 & 1 & 1 & -2 & -2 \\ 1 & 1 & 1 & 1 & -2 & -2 \\ 1 & 1 & 1 & 1 & -2 & -2 \\ 1 & 1 & 1 & 1 & -2 & -2 \\ -2 & -2 & -2 & -2 & 4 & 4 \\ -2 & -2 & -2 & -2 & 4 & 4 \end{bmatrix}, \text{ of rank 1.}$$

This shows that there is only one contrast for which the BLUE under this submodel could possibly exist, the contrast between treatments 1, 2, 3, 4 and treatments 5, 6. To see whether the BLUE for this contrast really exists, one may refer to Corollary 3.3(b). To make use of it, note that

$$(\mathbf{K}_0 - \tilde{\mathbf{K}}_0)(\mathbf{N}_0 - \tilde{\mathbf{N}}_0)' = \frac{2}{75}\begin{bmatrix} 34 & 34 & 34 & 34 & -68 & -68 \\ 34 & 34 & 34 & 34 & -68 & -68 \\ 5 & 5 & 5 & 5 & -10 & -10 \\ 5 & 5 & 5 & 5 & -10 & -10 \\ -39 & -39 & -39 & -39 & 78 & 78 \\ -39 & -39 & -39 & -39 & 78 & 78 \end{bmatrix}, \text{ of rank 1.}$$

Evidently, no (nonzero) linear combination of this matrix can be a linear combination of the matrix $(\mathbf{N}_0 - \tilde{\mathbf{N}}_0)'$. Also, it can be noted that

$$\mathbf{R}_0' = \frac{2}{5}\begin{bmatrix} 1 & 1 & 1 & 1 & -2 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & -1 & -1 & 2 & 2 \end{bmatrix}, \text{ of rank 1,}$$

from which

$$\mathbf{L}_0\mathbf{R}_0' = \frac{8}{75}\begin{bmatrix} 34 & 34 & 34 & 34 & -68 & -68 \\ 5 & 5 & 5 & 5 & -10 & -10 \\ -39 & -39 & -39 & -39 & 78 & 78 \end{bmatrix}, \text{ of rank 1.}$$

Certainly, no (nonzero) linear combination of the latter matrix can be a linear combination of the former. Thus, the conditions of Corollary 3.3(b) are not satisfied, and so there does not exist the BLUE under the inter-superblock submodel for any contrast, for that indicated above in particular.

Finally, if one is interested in estimating the general parameter mean, $\mathbf{c}'\boldsymbol{\tau} = n^{-1}\mathbf{r}'\boldsymbol{\tau}$, the BLUE of it is obtainable under the total-area submodel in the form $\mathbf{c}'\hat{\boldsymbol{\tau}} = \mathbf{s}'\boldsymbol{\Delta}\mathbf{y}_4$, where $\mathbf{s} = n^{-1}\mathbf{1}_v$, i.e. simply in the form $n^{-1}\mathbf{1}_n'\mathbf{y}$. This estimator is equal to $\mathbf{s}'\boldsymbol{\Delta}\mathbf{y}$. However, it is not the BLUE under the overall model (2.7), as the condition of Corollary 3.5(d) is not satisfied.

Example 4.2. Let the block design considered as Example 3.3 by Caliński and Kageyama (1991) [taken from Pearce (1983, p. 225)] for a 2^3 factorial structure of treatments, with the three factors denoted by X, Y and Z, be now redefined as a nested block design with two superblocks ($a = 2$), according to the partitioned incidence matrix $\mathbf{N} = [\mathbf{N}_1 : \mathbf{N}_2]$, where

Treatment	Superblock 1					Superblock 2			
1	0	1	1	1		0	0	0	0
X	0	0	0	0		2	1	1	1
Y	0	0	0	0		1	2	1	1
Z	0	0	0	0		1	1	2	1
XY	1	1	1	0		0	0	0	0
XZ	1	1	0	1		0	0	0	0
YZ	1	0	1	1		0	0	0	0
XYZ	0	0	0	0		1	1	1	2

$$\mathbf{N}_1 = \begin{bmatrix} 0&1&1&1\\0&0&0&0\\0&0&0&0\\0&0&0&0\\1&1&1&0\\1&1&0&1\\1&0&1&1\\0&0&0&0 \end{bmatrix} \quad \text{and} \quad \mathbf{N}_2 = \begin{bmatrix} 0&0&0&0\\2&1&1&1\\1&2&1&1\\1&1&2&1\\0&0&0&0\\0&0&0&0\\0&0&0&0\\1&1&1&2 \end{bmatrix}$$

(treatment 1 representing the combination of all three factors at the lower level, treatment X representing the combination of factor X at the upper level with Y and Z at the lower level, etc.). Each of the two superblocks is composed of four blocks ($b_1 = b_2 = 4$), of size 3 in the first superblock ($k_{1(1)} = k_{2(1)} = k_{3(1)} = k_{4(1)} = 3$) and of size 5 in the second ($k_{1(2)} = k_{2(2)} = k_{3(2)} = k_{4(2)} = 5$). From the incidence matrices $\mathbf{N}_h$, $h = 1, 2$, describing the component designs D_1 and D_2, the incidence matrix describing the design D is obtainable, as

$$\mathbf{R} = \begin{bmatrix} 3 & 0 \\ 0 & 5 \\ 0 & 5 \\ 0 & 5 \\ 3 & 0 \\ 3 & 0 \\ 3 & 0 \\ 0 & 5 \end{bmatrix}.$$

From this matrix, the vector of treatment replications for the whole design is $\mathbf{r} = [3, 5, 5, 5, 3, 3, 3, 5]'$, and the vector of superblock sizes is $\mathbf{n} = [12, 20]'$. Similarly as for the first example, it is assumed that this design is applied to available experimental units grouped into blocks and those joined into superblocks, all of them of conformable sizes, so that the appropriate threefold randomization can be performed.

For the estimation under the intra-block submodel one needs to examine the matrix $\mathbf{C}_1$. Here it is

$$\mathbf{C}_1 = \frac{2}{15}\begin{bmatrix} 15 & 0 & 0 & 0 & -5 & -5 & -5 & 0 \\ 0 & 27 & -9 & -9 & 0 & 0 & 0 & -9 \\ 0 & -9 & 27 & -9 & 0 & 0 & 0 & -9 \\ 0 & -9 & -9 & 27 & 0 & 0 & 0 & -9 \\ -5 & 0 & 0 & 0 & 15 & -5 & -5 & 0 \\ -5 & 0 & 0 & 0 & -5 & 15 & -5 & 0 \\ -5 & 0 & 0 & 0 & -5 & -5 & 15 & 0 \\ 0 & -9 & -9 & -9 & 0 & 0 & 0 & 27 \end{bmatrix}, \text{ of rank 6.}$$

It shows that due to the disconnectedness not all contrasts can be estimated under the intra-block submodel. In fact, the BLUEs under this submodel exist for contrasts giving the main effects and two factor interactions, but not for the contrast that gives the three factor interaction, i.e. not for the contrast represented by the vector $\mathbf{c} = [-1, 1, 1, 1, -1, -1, -1, 1]'$.

As to the estimation under the inter-block-intra-superblock submodel, it involves the matrix $\tilde{\mathbf{N}}_0 = [\mathbf{N}_{01}, \mathbf{N}_{02}]$, where

$$\mathbf{N}_{01} = \frac{1}{4}\begin{bmatrix} -3 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & -3 \\ 1 & 1 & -3 & 1 \\ 1 & -3 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{N}_{02} = \frac{1}{4}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & -1 & -1 & 3 \end{bmatrix}.$$

Hence,

$$
\mathbf{C}_2 = \frac{1}{60}\begin{bmatrix} 15 & 0 & 0 & 0 & -5 & -5 & -5 & 0 \\ 0 & 9 & -3 & -3 & 0 & 0 & 0 & -3 \\ 0 & -3 & 9 & -3 & 0 & 0 & 0 & -3 \\ 0 & -3 & -3 & 9 & 0 & 0 & 0 & -3 \\ -5 & 0 & 0 & 0 & 15 & -5 & -5 & 0 \\ -5 & 0 & 0 & 0 & -5 & 15 & -5 & 0 \\ -5 & 0 & 0 & 0 & -5 & -5 & 15 & 0 \\ 0 & -3 & -3 & -3 & 0 & 0 & 0 & 9 \end{bmatrix}, \text{ of rank 6.}
$$

By comparing $\mathbf{C}_2$ with $\mathbf{C}_1$ one can see that the same contrasts for which the BLUEs exist under the intra-block submodel can be considered for estimating under the inter-block-intra-superblock submodel. On account of Remark 3.1, the equality of block sizes within the superblocks implies that the BLUEs under this submodel really exist for all these contrasts. Thus, all the main effects and all the two-factor interactions receive BLUEs under both submodels.

Now, as to the estimation under the inter-superblock submodel, one has to take into account the matrix $\mathbf{C}_3$, which can be obtained (alternatively to the formula used in Example 4.1) from the formula $\mathbf{C}_3 = \mathbf{R}_0\mathbf{n}^{-\delta}\mathbf{R}_0'$ (see the proof of Theorem 3.3). Since here

$$
\mathbf{R}_0' = \frac{15}{8}\begin{bmatrix} 1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 \\ -1 & 1 & 1 & 1 & -1 & -1 & -1 & 1 \end{bmatrix},
$$

it follows that

$$
\mathbf{C}_3 = \frac{15}{32}\begin{bmatrix} 1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 \\ -1 & 1 & 1 & 1 & -1 & -1 & -1 & 1 \\ -1 & 1 & 1 & 1 & -1 & -1 & -1 & 1 \\ -1 & 1 & 1 & 1 & -1 & -1 & -1 & 1 \\ 1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 \\ 1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 \\ 1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 \\ -1 & 1 & 1 & 1 & -1 & -1 & -1 & 1 \end{bmatrix}, \text{ of rank 1.}
$$

Evidently, there is only one contrast which can be considered for estimation under the inter-superblock submodel. It is the contrast giving the three-factor interaction, that which cannot be estimated under the previous two submodels. As the columns of the incidence matrix $\mathbf{R}$ are linearly independent, the conditions of Corollary 3.4 are satisfied, on account of Remark 3.4(b). Thus, there really exists the BLUE of this contrast under the inter-superblock submodel. Also, it should be noted that the corresponding vector $\mathbf{s} = \mathbf{r}^{-\delta}\mathbf{c} = [-1/3,\ 1/5,\ 1/5,\ 1/5,\ -1/3,\ -1/3,\ -1/3,\ 1/5]'$ satisfies the equalities $\mathbf{C}_1\mathbf{s} = \mathbf{0}$, $\mathbf{C}_2\mathbf{s} = \mathbf{0}$ and $\mathbf{r}'\mathbf{s} = 0$, while $\mathbf{R}'\mathbf{s} \neq \mathbf{0}$, and that the condition (ii) of Theorem 2.2 is satisfied completely. Thus, by Corollary 3.5(c), $\mathbf{s}'\boldsymbol{\Delta}\mathbf{y}$ is the BLUE of the three-factor interaction, $\mathbf{c}'\boldsymbol{\tau}$, under the overall model (2.7).

Finally, since both parts of the condition (ii) of Theorem 2.2 are satisfied, $n^{-1}\mathbf{1}'_n\mathbf{y}$ is the BLUE of $n^{-1}\mathbf{r}'\boldsymbol{\tau}$ under the overall model (2.7), as it follows from Corollary 3.5(d).

5. Concluding remarks

The general theory concerning nested block designs (NB designs) presented in this paper reveals the possibilities of obtaining best linear unbiased estimators (BLUEs) for interesting treatment parametric functions either under the overall randomization model (2.7) or its submodels related to the various strata of the nested classification of available experimental units. Only in some special cases (Section 3.5) the BLUEs are obtainable under the overall model. Usually for an interesting function one can obtain the BLUE of it under one or more submodels. For contrasts of treatment parameters the submodels to be searched are the intra-block, inter-block-intra-superblock and inter-superblock submodels.

There may be contrasts for interest for which the BLUEs exist under all the three submodels, or only under one or two of them. This depends on the relations of the contrasts to the relevant C-matrices, and in case of the second and third submodel also on some additional conditions. In case of the second model, the additional condition relates the considered contrast to the departures of the incidence matrices $\mathbf{N}_h$ from relevant orthogonal structures, modified by possible block size inequalities within the component designs D_h, $h = 1, 2,\ldots,a$ [Corollary 3.1(b)]. One extreme case is when all D_h are orthogonal ($\mathbf{N}_h = n_h^{-1}\mathbf{r}_h\mathbf{k}'_h$). Then no contrasts can be estimated under the inter-block-intra-superblock submodel [Corollary 3.1(a)]. The other extreme case is when for any D_h for which $\mathbf{N}_h \neq n_h^{-1}\mathbf{r}_h\mathbf{k}'_h$ the block sizes are equal, i.e. $k_{1(h)} = k_{2(h)} = \ldots = k_{b_h(h)} = k_{(h)}$ (say). Then any contrast generated by $\mathbf{C}_2 = \tilde{\mathbf{N}}_0\mathbf{k}^{-\delta}\tilde{\mathbf{N}}'_0$ obtains the BLUE under this submodel (Remark 3.1). While the first extreme case would be rather uncommon, the second may happen quite often in practice, see, e.g., Example 4.1. Another quite common design case is that in which the columns of each $\mathbf{N}_h$ are linearly independent. In such a NB design the existence of BLUEs does not depend on the block sizes [Remark 3.2(b)]. To this class of NB designs, e.g., the α-designs (see, e.g., Patterson, Williams and Hunter, 1978) belong. As to the third submodel, the additional condition relates the considered contrast to the difference between the departure of the incidence matrix $\mathbf{N}$ from relevant orthogonal structure and such departures of the $\mathbf{N}_h$ incidence matrices, this being modified by corresponding possible block size inequalities, and also to the departure of the incidence matrix $\mathbf{R}$ from relevant orthogonal structure modified by possible inequalities among superblock sizes [Corollary 3.3(b)]. Again, two extreme cases can be visualized: the first, when the design D is orthogonal, the second, when all block sizes are equal and all superblock sizes are equal. In the first of these cases no contrast can be estimated under the inter-superblock submodel [Corollary 3.3(a)], in the second, any contrast generated by $\mathbf{C}_3 = \mathbf{R}_0\mathbf{n}^{-\delta}\mathbf{R}_0'$ will obtain the BLUE under this submodel. Here, in fact, the first extreme case can be considered as the most common. The resolvable block designs are usually constructed in that way. On the other hand some factorial experiments may be designed in such a way that $\mathbf{R} \neq n^{-1}\mathbf{rn}'$, see, e.g., Example 4.2. If in such a case the columns of the incidence matrix $\mathbf{R}$ are linearly independent (as in this example), then for any contrast generated by $\mathbf{C}_3$ the BLUE exists under this submodel, independently of the block and superblock sizes [Remark 3.4(b)].

The total-area submodel provides BLUEs for the general parametric mean, or its multiplicities, only. This function cannot be estimated under any other submodel. It obtains the BLUE under the overall model if and only if the block sizes are constant within any connected subdesign of D^*, and the sizes of the superblocks are constant within any of the connected subdesigns of D [Corollary 3.5(d)].

Finally it should be noted that if a contrast is estimated under more than one submodel, then it is desirable to combine the obtained information on the contrast from the relevant strata. This is not discussed in this paper as it needs separate consideration.

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O randomizacyjnej teorii doświadczeń w układach o blokach zagnieżdżonych

Streszczenie

W pracy przedstawiono ogólny model randomizacyjny dla doświadczeń w układach o blokach zagnieżdżonych, to znaczy dwuwarstwowych, oraz podano warunki otrzymywania najlepszych liniowych estymatorów nieobciążonych w tym modelu. Ponieważ okazuje się, że warunki te są bardzo ograniczające, rozważane jest rozłożenie tego modelu na cztery odpowiednie podmodele warstwowe. Znaleziono warunki otrzymywania najlepszych liniowych estymatorów nieobciążonych w tych podmodelach. Ponadto pokazano, pod jakimi warunkami najlepszy liniowy estymator funkcji liniowej parametrów obiektowych otrzymany w jednym z podmodeli jest jednocześnie takim estymatorem w modelu całościowym. Dwa dyskutowane przykłady ilustrują zastosowanie podanej teorii. Na zakończenie podano kilka wniosków ogólnych dotyczących rozważanych układów doświadczalnych.

Słowa kluczowe: najlepsza liniowa estymacja nieobciążona, układy o blokach zagnieżdżonych, analiza wewnątrzblokowa, analiza międzyblokowo-wewnątrzsuperblokowa, analiza międzysuperblokowa, model randomizacyjny.